Statistical mechanics of relativistic one-dimensional self-gravitating systems

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We consider the statistical mechanics of a general relativistic one-dimensional self-gravitating system. The system consists of N particles coupled to lineal gravity and can be considered as a model of N relativistically interacting sheets of uniform mass. The partition function and one-particle distribution functions are computed to leading order in $1/c$ where c is the speed of light; as $c\to\infty$ results for the nonrelativistic one-dimensional self-gravitating system are recovered. We find that relativistic effects generally cause both position and momentum distribution functions to become more sharply peaked, and that the temperature of a relativistic gas is smaller than its nonrelativistic counterpart at the same fixed energy. We consider the large-N limit of our results and compare this to the nonrelativistic case.

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I. INTRODUCTION

One-dimensional systems of N particles mutually interacting through gravitational forces have been of interest in astrophysics for more than three decades. While used primarily as prototypes for the behavior of gravity in higher dimensions, one-dimensional self-gravitating systems (OGS's) also conjectured to approximate the behavior of some physical systems in three spatial dimensions. These include the dynamics of stars in a direction orthogonal to the plane of a highly flattened galaxy [1] and the collisions of flat parallel domain walls [2] (i.e., sheets of stress energy [1]) moving in directions perpendicular to their surfaces. Furthermore, very long lived core-halo structures in the OGS phase space are known to exist, reminiscent of structures observed in globular clusters, in which a dense massive core in near equilibrium is surrounded by a halo of stars with high kinetic energy that interact only weakly with the core [3].

The statistical properties of the OGS are particularly intriguing. Despite extensive study, many unanswered questions remain. For example, it is not clear if the OGS can attain a true equilibrium state from arbitrary initial conditions. Its ergodic and equipartition properties are still not well understood. This is primarily because the particle interactions of the OGS (as with any self-gravitating system) are attractive and cumulatively long range, in strong contrast to typical thermodynamic systems for which such interactions are repulsive and short range. For the OGS the macroscopic dynamics does not decouple from the microscopic dynamics, and the usual thermodynamic analysis does not apply.

However, there are some established features of the OGS. Rybicki [1] derived in closed form the single-particle distribution function in both the canonical and microcanonical ensembles. In the large-N limit these distribution functions reduce to the isothermal solution of the Vlasov equation.

All studies to date have neglected relativistic effects. This limitation is understandable since no relativistic N-particle Hamiltonian was available for analysis. However, this situation changed recently when a prescription for obtaining the Hamiltonian for a relativistic one-dimensional self-gravitating system (ROGS) was given by Mann and Ohta [4]. This Hamiltonian can be rigorously derived from a generally covariant system coupling relativistic gravity in one spatial dimension (i.e., a $1+1$ dimensional theory of gravity [5]) to N point particles. In the nonrelativistic limit, the Hamiltonian reduces to that of the OGS. Although not available in closed form, the Hamiltonian can be obtained as a series expansion in inverse powers of the speed of light c to arbitrary order.

We consider in this paper the one-particle distribution function for the ROGS. Our work here is a natural extension of previous work on the N-body problem in relativistic gravity. In three spatial dimensions an exact solution to this problem is known for pure Newtonian gravity (and a series solution has been constructed for arbitrary N). In the general theory of relativity dissipation of energy in the form of gravitational radiation has obstructed progress toward obtaining exact solutions to the N-body problem even when $N=2$. However, for the ROGS an exact solution to the two-body problem was recently obtained [6], and generalizations including a cosmological constant and/or charge subsequently followed [7–9]. These solutions include both an explicit expression for the proper separation of the two bodies as a function of time and an explicit expression for the Hamiltonian for the two-body ROGS as a function of the proper separation and the center-of-inertia momentum of the bodies.

Encouraged by these results, we here make an attempt to understand the basic features of the N-body ROGS. We shall recapitulate the canonical formalism used in Ref. [4] to derive the Hamiltonian for the N-body ROGS. We then compute the partition function and canonical distribution functions. Using an integral transform we then calculate the microcanonical distribution functions. All results are in closed form to leading order in $1/c$. We consider the limit of large N and compare the ROGS and the OGS. We close with a few remarks. Lengthy intermediate calculations are confined to the Appendix.

We emphasize that, although we begin with a generally covariant minimally coupled multiparticle action, we do not

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have a fully general relativistic statistical mechanics. Such a formulation is not yet within reach, primarily due to interpretational issues associated with retardation effects, the choice of time coordinate, and the role of energy. However, although we do not have an explicit form for the exact N-body ROGS Hamiltonian, we do have a systematic means of computing it to any desired order in $1/c$. We compute all quantities to leading order in this parameter, interpreting the results in the context of the flat background about which the expansion is carried out.

II. CANONICAL N-PARTICLE HAMILTONIAN OF THE ROGS

The OGS Hamiltonian is

$$H=\sum_a \frac{p_a^2}{2m_a}+\pi G\sum_a \sum_b m_a m_b|z_a-z_b| =\sum_a \frac{p_a^2}{2m_a}+2\pi G\sum_{a>b} m_a m_b|z_a-z_b|, \tag{1}$$

where the summation is over all N particles, located at positions z_a along the spatial axis. The potential term straightforwardly follows upon solving the Newtonian equation

$$\nabla^2\varphi=4\pi G\rho, \tag{2}$$

in one spatial dimension, where $\rho=\Sigma_a m_a\delta(x-z_a)$ is the mass density of the N point particles. Our task in this section is to find a prescription for obtaining a relativistic generalization of Eq. (1).

The Hamiltonian for the ROGS that we use is that of a $(1+1)$-dimensional theory (a lineal gravity theory) that models $(3+1)$-dimensional general relativity in that it sets the Ricci scalar equal to the trace of the stress energy of prescribed matter fields and sources. Hence matter governs the evolution of space-time curvature that reciprocally governs the evolution of matter [5]. We refer to this theory as $R=T$ theory. Apart from being able to model a number of textbook scenarios in general relativity [10], it has the attractive feature of having a consistent Newtonian limit [5]. This limit, essential for our purposes, is problematic in a generic $(1+1)$-dimensional theory of gravity [11].

Since the Einstein action is a topological invariant in $(1+1)$ dimensions, a scalar (dilaton) field must be included in the action [12]. Its coupling to the curvature is chosen so that only the trace of the stress energy of matter (N point particles here) is set equal to the Ricci scalar. This action will form the basis for the ROGS we consider. Upon canonical reduction of the action [4], the ROGS Hamiltonian is given in terms of a spatial integral of the second derivative of the dilaton field, which is a function of the coordinates and momenta of the particles and is determined from the constraint equations.

The action integral for the gravitational field coupled to N point particles is

$$I=\int d^2x\left[\frac{1}{2\kappa}\sqrt{-g}g^{\mu\nu}\left\{\Psi R_{\mu\nu}+\frac{1}{2}\nabla_\mu\Psi\nabla_\nu\Psi\right\} +\sum_a\int d\tau_a\left\{-m_a\left(-g_{\mu\nu}(x)\frac{dz_a^\mu}{d\tau_a}\frac{dz_a^\nu}{d\tau_a}\right)^{1/2}\right\} \times\delta^2(x-z_a(\tau_a))\right], \tag{3}$$

where Ψ is the dilaton field, $g_{\mu\nu}$ and g are the metric and its determinant, R is the Ricci scalar, and τ_a is the proper time of ath particle whose mass is m_a, with $\kappa=8\pi G/c^4$. We use ∇_μ to denote the covariant derivative associated with $g_{\mu\nu}$.

The field equations derived from the action (3) are

$$R-g^{\mu\nu}\nabla_\mu\nabla_\nu\Psi=0, \tag{4}$$

$$\frac{1}{2}\nabla_\mu\Psi\nabla_\nu\Psi-\frac{1}{4}g_{\mu\nu}\nabla^\lambda\Psi\nabla_\lambda\Psi+g_{\mu\nu}\nabla^\lambda\nabla_\lambda\Psi-\nabla_\mu\nabla_\nu\Psi =\kappa T_{\mu\nu}+\frac{1}{2}g_{\mu\nu}\Lambda, \tag{5}$$

$$\frac{d}{d\tau_a}\left\{g_{\mu\nu}(z_a)\frac{dz_a^\nu}{d\tau_a}\right\}-\frac{1}{2}g_{\nu\lambda,\mu}(z_a)\frac{dz_a^\nu}{d\tau_a}\frac{dz_a^\lambda}{d\tau_a}=0, \tag{6}$$

where

$$T_{\mu\nu}=\sum_a m_a\int d\tau_a\frac{1}{\sqrt{-g}}g_{\mu\sigma}g_{\nu\rho}\frac{dz_a^\sigma}{d\tau_a}\frac{dz_a^\rho}{d\tau_a}\delta^2(x-z_a(\tau_a)) \tag{7}$$

is the stress energy due to the point masses. Equation (5) guarantees the conservation of $T_{\mu\nu}$. Inserting the trace of Eq. (5) into Eq. (4) yields

$$R-\Lambda=\kappa T^\mu{}_\mu. \tag{8}$$

Equations (5), (6), and (8) form a closed system of equations for gravity and matter.

In order to obtain the Hamiltonian in canonical form, we first decompose the scalar curvature in terms of the extrinsic curvature K via

$$\sqrt{-g}R=-2\partial_0(\sqrt{\gamma}K)+2\partial_1[(N_1K-\partial_1N_0)/\sqrt{\gamma}], \tag{9}$$

where the metric is

$$ds^2=-N_0^2dt^2+\gamma\left(dx+\frac{N_1}{\gamma}dt\right)^2, \tag{10}$$

with $K=(2N_0\gamma)^{-1}(2\partial_1N_1-\gamma^{-1}N_1\partial_1\gamma-\partial_0\gamma)$, so that $\gamma=g_{11}$, $N_0=(-g^{00})^{-1/2}$ and $N_1=g_{10}$. Rewriting the action (3) in first-order form yields

$$I=\int d^2x\Bigg\{\sum_a p_a\dot{z}_a\delta(x-z_a(t))+\pi\dot{\gamma}+\Pi\dot{\Psi}+N_0R^0+N_1R^1\Bigg\}, \tag{11}$$

where π and Π are the respective conjugate momenta to γ and Ψ. Here

$$R^0=-\kappa\sqrt{\gamma}\gamma\pi^2+2\kappa\sqrt{\gamma}\pi\Pi+\frac{1}{4\kappa\sqrt{\gamma}}(\Psi')^2-\frac{1}{\kappa}\left(\frac{\Psi'}{\sqrt{\gamma}}\right)'-\sum_a\sqrt{\frac{p_a^2}{\gamma}+m_a^2}\,\delta(x-z_a(t)),$$

$$R^1=\frac{\gamma'}{\gamma}\pi-\frac{1}{\gamma}\Pi\Psi'+2\pi'+\sum_a\frac{p_a}{\gamma}\delta(x-z_a(t)) \tag{12}$$

with the overdot and prime denoting ∂_0 and ∂_1, respectively.

Variation of the action (11) yields the set of equations

$$\dot{\pi}+N_0\Bigg\{\frac{3\kappa}{2}\sqrt{\gamma}\pi^2-\frac{\kappa}{\sqrt{\gamma}}\pi\Pi+\frac{1}{8\kappa\sqrt{\gamma}\gamma}(\Psi')^2-\sum_a\frac{p_a^2}{2\gamma^2\sqrt{\frac{p_a^2}{\gamma}+m_a^2}}\delta(x-z_a(t))\Bigg\}+N_1\Bigg\{-\frac{1}{\gamma^2}\Pi\Psi'+\frac{\pi'}{\gamma}+\sum_a\frac{p_a}{\gamma^2}\delta(x-z_a(t))\Bigg\}+N_0'\frac{1}{2\kappa\sqrt{\gamma}\gamma}\Psi'+N_1'\frac{\pi}{\gamma}=0, \tag{13}$$

$$\dot{\gamma}-N_0(2\kappa\sqrt{\gamma}\gamma\pi-2\kappa\sqrt{\gamma}\Pi)+N_1\frac{\gamma'}{\gamma}-2N_1'=0, \tag{14}$$

$$R^0=0, \tag{15}$$

$$R^1=0, \tag{16}$$

$$\dot{\Pi}+\partial_1\left(-\frac{1}{\gamma}N_1\Pi+\frac{1}{2\kappa\sqrt{\gamma}}N_0\Psi'+\frac{1}{\kappa\sqrt{\gamma}}N_0'\right)=0, \tag{17}$$

$$\dot{\Psi}+N_0(2\kappa\sqrt{\gamma}\pi)-N_1\left(\frac{1}{\gamma}\Psi'\right)=0. \tag{18}$$

$$\dot{p}_a+\frac{\partial N_0}{\partial z_a}\sqrt{\frac{p_a^2}{\gamma}+m_a^2}-\frac{N_0}{2\sqrt{\frac{p_a^2}{\gamma}+m_a^2}}\frac{p_a^2}{\gamma^2}\frac{\partial\gamma}{\partial z_a}-\frac{\partial N_1}{\partial z_a}\frac{p_a}{\gamma}+N_1\frac{p_a}{\gamma^2}\frac{\partial\gamma}{\partial z_a}=0, \tag{19}$$

$$\dot{z}_a-N_0\frac{\frac{p_a}{\gamma}}{\sqrt{\frac{p_a^2}{\gamma}+m_a^2}}+\frac{N_1}{\gamma}=0. \tag{20}$$

All metric components (N_0,N_1,γ) in Eqs. (19) and (20) are evaluated at the point $x=z_a$, where

$$\frac{\partial f}{\partial z_a}\equiv\left.\frac{\partial f(x)}{\partial x}\right|_{x=z_n}.$$

The quantities N_0 and N_1 are Lagrange multipliers that yield the constraint equations (15) and (16). The above set of equations can be proved to be equivalent to the set of equations (4), (5), and (6) [4].

An examination of the generator of space and time transformations [4,6] indicates that we find that we can consistently choose the coordinate conditions

$$\gamma=1\quad\text{and}\quad\Pi=0, \tag{21}$$

upon which the action (11) reduces to

$$I=\int d^2x\Bigg\{\sum_a p_a\dot{z}_a(x-z_a)-\mathcal{H}\Bigg\}, \tag{22}$$

after elimination of the constraints, where

$$H=\int dx\,\mathcal{H}=-\frac{1}{\kappa}\int\Delta\Psi \tag{23}$$

is the Hamiltonian for the ROGS.

The field Ψ is no longer arbitrary, but is instead a function of z_a and p_a that is determined by solving the constraints, which are now

$$\Delta\Psi-\frac{1}{4}(\Psi')^2+\kappa^2\pi^2+\kappa\sum_{a=1}^N\sqrt{p_a^2+m_a^2}\,\delta(x-z_a)=0, \tag{24}$$

$$2\pi'+\sum_{a=1}^N p_a\delta(x-z_a)=0, \tag{25}$$

once the coordinate conditions (21) are imposed. Equation (24) is an energy-balance equation which states that the energy of the particles plus the (negative) gravitational energy must vanish. Equation (25) states that the total momentum of the gravitational field and the particles must vanish. The consistency of this canonical reduction was proved in Ref. [4] by showing that the canonical equations of motion derived from the reduced Hamiltonian (23) are identical with the Eqs. (19) and (20).

The choice of coordinates (21) is the $(1+1)$-dimensional analog of that made in the standard Arnowitt-Deser-Misner decomposition in $(3+1)$ dimensions [4,6]. It has the advantage that it renders the Hamiltonian (23) explicitly time independent; it is implicitly time dependent insofar as it is a function only of the coordinates and momenta of the N particles in the system, each of which is time dependent. This Hamiltonian has been shown to be equivalent to that obtained using the Noether theorem associated with diffeomorphism symmetry [13]. For a single particle, it is straightforward to show that the solution to the above system of equations yields a metric that is asymptotically Rindlerian (flat space in accelerated coordinates) on either side of the mass [4]. A set of N particles localized within a finite region on the line will yield a metric with similar asymptotic behavior, since all delta-function contributions to the equations of motion vanish at large distances.

We turn next to an evaluation of the Hamiltonian (23).

III. COMPUTATION OF THE ROGS HAMILTONIAN

Although the constraint equations are straightforward to solve in the regions between the particles, the matching conditions of these solutions at the juncture of the particles are quite nontrivial. For the two-body ROGS their enforcement yields an equation that determines the Hamiltonian in terms of the remaining degrees of freedom of the system. While this procedure holds in principle for the N-body ROGS, we have not found a tractable means of obtaining an analogous determining equation for the Hamiltonian.

However, it is possible to straightforwardly and rigorously construct approximation schemes for computing the ROGS Hamiltonian for N particles. For example, the postlinear approximation is an expansion of the Hamiltonian in powers of the gravitational coupling κ, obtained by writing

$$\Psi=\kappa\Psi^{(1)}+\kappa^2\Psi^{(2)}+\cdots, \tag{26}$$

$$\chi=\chi^{(0)}+\kappa\chi^{(1)}+\cdots, \tag{27}$$

where χ is defined by $\chi'\equiv\pi$. Insertion of these expansions into Eqs. (24),(25) yields

$$\begin{aligned}H^{(2)}=&\sum_a \sqrt{p_a^2+m_a^2}+\frac{\kappa}{8}\sum_a\sum_b \left(\sqrt{p_a^2+m_a^2}\sqrt{p_b^2+m_b^2}-p_ap_b\right)|r_{ab}|+\frac{\epsilon\kappa}{8}\sum_a\sum_b\left(\sqrt{p_a^2+m_a^2}p_b-p_a\sqrt{p_b^2+m_b^2}\right)r_{ab}\\&+\frac{1}{4}\left(\frac{\kappa}{4}\right)^2\Bigg\{\sum_a\sqrt{p_a^2+m_a^2}\left[\sum_b p_b|r_{ab}|+\epsilon\sum_b\sqrt{p_b^2+m_b^2}r_{ab}\right]^2-\sum_a p_a\left[\sum_b p_b|r_{ab}|+\epsilon\sum_b\sqrt{p_b^2+m_b^2}r_{ab}\right]\\&\times\left[\sum_c\sqrt{p_c^2+m_c^2}|r_{ac}|+\epsilon\sum_c p_cr_{ac}\right]+\sum_a\sum_b\left[\sqrt{p_a^2+m_a^2}\sqrt{p_b^2+m_b^2}|r_{ab}|-\epsilon p_a\sqrt{p_b^2+m_b^2}r_{ab}\right]\left[\sum_c\sqrt{p_c^2+m_c^2}|r_{bc}|\right.\\&\left.+\epsilon\sum_c p_cr_{bc}\right]-\sum_a\sum_b\left[\sqrt{p_a^2+m_a^2}p_b|r_{ab}|-\epsilon p_ap_br_{ab}\right]\left[\sum_c p_c|r_{bc}|+\epsilon\sum_c\sqrt{p_c^2+m_c^2}r_{bc}\right]\Bigg\},\end{aligned} \tag{28}$$

upon insertion into Eq. (23), where $r_{ab}=z_a-z_b$ is the relative separation between particles a and b. It can be shown that the solutions to Eqs. (24),(25) must satisfy the boundary condition $\Psi^2-4\kappa^2\chi^2=0$ in the regions $|x|\gg|z_a|$ in order for the Hamiltonian to be finite [4].

The κ expansion is appropriate for describing relativistic fast motion of the particles and can be carried out to any desired order. However, to compare the ROGS from $R=T$ theory with the OGS, we turn to the post-Newtonian expansion, which is an expansion of the Hamiltonian in powers of c^{-1}. Since both p_a^2/m_a^2 and $\sqrt{\kappa}$ are of the order of c^{-2} all terms up to the order of c^{-4} are included in the postlinear Hamiltonian (28). The post-Newtonian Hamiltonian to this order is, therefore [4],

$$\begin{aligned}H=&\sum_{a=1}^N m_ac^2+\sum_{a=1}^N\frac{p_a^2}{2m_a}+2\pi G\sum_{a>b}^N m_am_b|r_{ab}|\\&-\frac{1}{c^2}\sum_{a=1}^N\frac{p_a^4}{8m_a^3}+\frac{\pi G}{c^2}\sum_{a=1}^N\sum_{b=1}^N m_a\frac{p_b^2}{m_b}|r_{ab}|\\&-\frac{2\pi G}{c^2}\sum_{a>b}^N p_ap_b|r_{ab}|+\left(\frac{\pi G}{c}\right)^2\sum_{a=1}^N\sum_{b=1}^N\sum_{c=1}^N\\&\times m_am_bm_c[|r_{ab}||r_{ac}|-r_{ab}r_{ac}]+\cdots,\end{aligned} \tag{29}$$

where the explicit powers of c have been restored.

The first term in Eq. (29) is the total rest energy of the particles, and the second two terms are the OGS Hamiltonian (1). The remaining terms are all relativistic corrections to the OGS to order c^{-2}. The first of these corrections is a special-relativistic one, whereas the remaining corrections are due to relativistic gravity in one spatial dimension. Note that gravity not only modifies the potential to a quadratic form, but also includes couplings between particle momenta and their positions. These features—modifications of the distance behavior of the potential and position-momentum couplings—are fully analogous to those in general relativity in three spatial dimensions.

We next find the equations of motion for the position of the ath particle. This follows straightforwardly from Hamilton's principle. We have

$$\dot{z}_a=\frac{\partial H}{\partial p_a}$$
$$=\frac{p_a}{m_a}-\frac{1}{c^2}\left[\frac{p_a^3}{2m_a^3}-\frac{\kappa}{4}\sum_{b=1}^{N} m_b\frac{p_a}{m_a}|r_{ab}|+\frac{\kappa}{4}\sum_{b=1}^{N} p_b|r_{ab}|\right], \tag{30}$$

and

$$\dot{p}_a=-\frac{\partial H}{\partial z_a}$$
$$=-2\pi G\sum_{b=1}^{N} m_a m_b\varepsilon_{ab}-\frac{1}{c^2}\Bigg[\pi G\sum_{b=1}^{N}\left(m_a\frac{p_b^2}{m_b}+m_b\frac{p_a^2}{m_a}\right)\times\varepsilon_{ab}-2\pi G\sum_{b=1}^{N} p_a p_b\varepsilon_{ab}+2(\pi G)^2\times\sum_{b=1}^{N}\sum_{c=1}^{N} m_a m_b m_c[\varepsilon_{ab}|r_{ac}|+\varepsilon_{ab}|r_{bc}|-r_{ab}]\Bigg], \tag{31}$$

where

$$\varepsilon_{ab}=\begin{cases}1 & z_a>z_b\\ -1 & z_a<z_b.\end{cases}$$

We can solve Eq. (30) for p_a,

$$p_a=m_a\dot{z}_a+\frac{1}{c^2}\left[m_a\frac{\dot{z}_a^3}{2}+2\pi G\sum_{b=1}^{N}[(m_b m_a\dot{z}_a-m_a m_b\dot{z}_b)|r_{ab}|]\right]+\cdots, \tag{32}$$

and then insert this into Eq. (31) for $\dot{p}_a$,

$$m_a\ddot{z}_a+\frac{1}{c^2}\Bigg[m_a\ddot{z}_a\frac{3\dot{z}_a^2}{2}-2\pi G\sum_{b=1}^{N}[(m_b m_a\ddot{z}_a-m_a m_b\ddot{z}_b)|r_{ab}|+(m_b m_a\dot{z}_a-m_a m_b\dot{z}_b)\varepsilon_{ab}(\dot{z}_a-\dot{z}_b)]\Bigg]$$
$$=-2\pi G\sum_{b=1}^{N} m_a m_b\varepsilon_{ab}-\frac{1}{c^2}\Bigg[\pi G\sum_{b=1}^{N}(m_a m_b[\dot{z}_b^2+\dot{z}_a^2])\times\varepsilon_{ab}-2\pi G\sum_{b=1}^{N} m_a m_b\dot{z}_a\dot{z}_b\varepsilon_{ab}+2(\pi G)^2\times\sum_{b=1}^{N}\sum_{c=1}^{N} m_a m_b m_c[\varepsilon_{ab}|r_{ac}|+\varepsilon_{ab}|r_{bc}|-r_{ab}]\Bigg], \tag{33}$$

which simplifies to

$$m_a\ddot{z}_a=-2\pi G\sum_{b=1}^{N} m_a m_b\varepsilon_{ab}+\frac{\pi G}{c^2}\sum_{b=1}^{N}(m_a m_b\{3\dot{z}_a^2+[\dot{z}_b-\dot{z}_a]^2\}\varepsilon_{ab})-\left(\frac{2\pi G}{c}\right)^2\sum_{b=1}^{N}\sum_{c=1}^{N} m_a m_b m_c(\varepsilon_{ac}-\varepsilon_{bc})|r_{ab}|-2\left(\frac{\pi G}{c}\right)^2\sum_{b=1}^{N}\sum_{c=1}^{N} m_a m_b m_c[\varepsilon_{ab}|r_{ac}|+\varepsilon_{ab}|r_{bc}|-r_{ab}], \tag{34}$$

upon an iterative substitution of $\ddot{z}_a$ in powers of c^{-1}. These equations of motion reduce to those of the OGS in the limit $c\rightarrow\infty$, and may be shown to be equivalent to the geodesic equations to this order [4].

As noted previously, we shall interpret the ROGS Hamiltonian (29) in a post-Newtonian flat-space context. This of course sidesteps the deeper interpretational issues involved in the development of a fully generally relativistic statistical mechanics. However post-Newtonian flat-space interpretations of $(3+1)$-dimensional general relativity have been of enormous use in understanding how relativistic effects modify Newtonian physics (e.g., perihelion precession, bending of light). The approximations we employ here are no more severe, and so a study of the physics associated with the Hamiltonian (29) should afford us insight as to how general-relativistic effects modify statistical systems. For example, retardation effects are accounted for by considering the system as a set of particles moving in momentum-dependent potentials.

We wish to investigate the intrinsic structure of the system described by the Hamiltonian (29). However, because of the translation invariance of the system, two phase-space degrees of freedom are redundant, and so must be factored out; otherwise certain average properties such as density would be uniform throughout space.

Using Eq. (31), it is straightforward to show that

$$\sum_{a=1}^{N} \dot{p}_a = 0, \tag{35}$$

and [using also Eq. (30)] that the Hamiltonian is time independent. This means that we can perform the phase-space integration subject to the constraint

$$\overline{p} = 0 \quad \text{where} \quad \overline{p} \equiv \sum_{a=1}^{N} p_a, \tag{36}$$

since we can choose a frame of reference in which the center of inertia is constant.

Removing the redundant position degree-of-freedom is somewhat more delicate. Although the system is invariant under the translation $z_a \to z_a + \hat{z}$, Eq. (32) implies

$$\overline{p} \equiv \frac{1}{N} \sum_{a=1}^{N} p_a = \frac{1}{N} \sum_{a=1}^{N} \left[m_a \dot{z}_a \left(1 + \frac{1}{2} \frac{\dot{z}_a^2}{c^2} \right) \right], \tag{37}$$

which cannot be written as a total time derivative. Physically, the center of inertia is the relativistically well-defined concept, whereas the center of mass is not. However, we can deal with this problem by inserting a factor of unity in all phase-space averages in the form

$$\int_{-\infty}^{\infty} dL \; \delta(\overline{z} - L),$$

where $\overline{z} \equiv (1/M) \Sigma_{a=1}^{N} m_a z_a$, with $M = \Sigma_{a=1}^{N} m_a$. If the L dependence of any integral trivially factors out (or can be removed by a shift of variable in the integrand), then we regard the remaining quantity as the physically relevant one to describe the system.

For a canonical ensemble, all phase-space averages are carried out with a weighting function $\exp|-\beta H|$, where $k_B T = \beta^{-1}$ is the temperature multiplied by Boltzmann's constant k_B. For the microcanonical ensemble, an additional constraint of fixed total energy

$$H(z_a, p_a) = E,$$

must be included, consistent with the time independence of the Hamiltonian (29). Since the system is in momentum isolation, it is difficult to see how it can be in energy contact with a heat bath, and so the physical relevance of the canonical ensemble is somewhat unclear. However, an evaluation of quantities within the canonical ensemble is instructive in its own right and is a necessary preliminary to computing quantities in the more realistic microcanonical ensemble, and so we include it in the present discussion.

Henceforth we set $m_a = m$, so that $M = Nm$.

IV. THE CANONICAL ENSEMBLE

We consider in this section the relativistic corrections to the canonical one-particle distribution function $f_c^R(p,z)$, which is defined to be the phase-space average of the quantity

$$\frac{1}{N} \sum_a \delta(z - z_a) \delta(p - p_a), \tag{38}$$

weighted by $\exp(-\beta H)$ with the constraint $\hat{p} = 0$. Hence

$$f_c^R(p,z) = \frac{1}{\mathcal{Z} N!} \int \int d\mathbf{p} \, d\mathbf{z} \; \delta(\overline{p}) \int_{-\infty}^{\infty} dL \; \delta(\overline{z} - L) \times \exp(-\beta H) N^{-1} \sum_a \delta(z - z_a) \delta(p - p_a), \tag{39}$$

where

$$\mathcal{Z} = \frac{1}{N!} \int \int d\mathbf{p} \, d\mathbf{z} \; \delta(\overline{p}) \int_{-\infty}^{\infty} dL \; \delta(\overline{z} - L) \exp(-\beta H) \tag{40}$$

is the partition function and where the second line in Eq. (39) follows from the indistinguishability of the particles. Note that a shift of integration variable

$$z_a' = z_a + L,$$

renders the partition function in the form

$$\mathcal{Z} = \frac{1}{N!} \int_{-\infty}^{\infty} dL \int \int d\mathbf{p} \, d\mathbf{z}' \; \delta(\overline{p}) \delta(\overline{z}') \exp(-\beta H), \tag{41}$$

where the L dependence is seen to trivially factor out. It can, therefore, be dropped (along with the prime notation) from further consideration in the evaluation of $\mathcal{Z}$. Similarly the single-particle distribution function becomes

$$f_c^R(p,z) = \frac{1}{\mathcal{Z} N!} \int_{-\infty}^{\infty} dL \int \int d\mathbf{p} \, d\mathbf{z} \; \delta(\overline{p}) \delta(\overline{z}') \times \exp(-\beta H) N^{-1} \sum_a \delta(z - L - z_a') \delta(p - p_a), \tag{42}$$

which is of the form $\int_{-\infty}^{\infty} dL \, f_c'^R(p, z - L)$. We, therefore, regard $f_c'^R(p, z - L)$ as the physically relevant quantity, where

$$f_c'^R(p,z) = \frac{1}{\mathcal{Z} N!} \int \int d\mathbf{p} \, d\mathbf{z} \; \delta(\overline{p}) \delta(\overline{z}') \times \exp(-\beta H) N^{-1} \sum_a \delta(z - z_a') \delta(p - p_a), \tag{43}$$

and where the primes will hencefore be dropped.

Unlike the nonrelativistic case, neither the partition function nor $f_c^R(p,z)$ are separable. We proceed by first evaluating the partition function.

We first write the Hamiltonian (29) in the following form:

$$H = Mc^2 + H_0 + \frac{1}{c^2} H_R, \tag{44}$$

$$H_0 = \sum_{a=1}^{N} \frac{p_a^2}{2m} + 2\pi G m^2 \sum_{a>b}^{N} |r_{ab}|, \tag{45}$$

$$H_R = -\sum_{a=1}^{N} \frac{p_a^4}{8m^3} + \pi G \sum_{a=1}^{N}\sum_{b=1}^{N} p_b^2 |r_{ab}| - 2\pi G \sum_{a>b}^{N} p_a p_b |r_{ab}| + (\pi G)^2 \sum_{a=1}^{N}\sum_{b=1}^{N}\sum_{c=1}^{N} m^3 [|r_{ab}||r_{ac}| - r_{ab} r_{ac}], \tag{46}$$

so that

$$\exp(-\beta H) = e^{-\beta M c^2} e^{-\beta H_0}\left(1 - \frac{\beta}{c^2} H_R\right) + O\left(\frac{\beta^2}{c^4}\right), \tag{47}$$

which is valid to the order in which we are working. Writing

$$\delta(\overline{p}) = \frac{1}{2\pi}\int dk \exp\left[ik \sum_{a=1}^{N} p_a\right], \tag{48}$$

we have

$$\mathcal{Z} = \frac{c^{-\beta M c^2}}{N!}\int d\mathbf{z}\, \delta(\overline{z}) \exp\left(-2\pi G \beta m^2 \sum_{a>b}^{N} |r_{ab}|\right) \times \int \frac{dk}{2\pi}\int d\mathbf{p} \exp\left(ik\sum_{a=1}^{N} p_a - \beta \sum_{a=1}^{N}\frac{p_a^2}{2m}\right) \times \left(1 - \frac{\beta}{c^2} H_R\right). \tag{49}$$

A. The partition function

Consider first the integral

$$\int \frac{dk}{2\pi}\int d\mathbf{p} \exp\left(ik\sum_{a=1}^{N} p_a - \beta\sum_{a=1}^{N}\frac{p_a^2}{2m}\right)\left(1 - \frac{\beta}{c^2} H_R\right), \tag{50}$$

which has integrals that are at most quartic in the momenta. Straightforward Gaussian integration yields

$$\int \frac{dk}{2\pi}\int d\mathbf{p} \exp\left(ik\sum_{c=1}^{N} p_c - \beta\sum_{c=1}^{N}\frac{p_c^2}{2m}\right) \times \begin{Bmatrix} 1 \\ p_a p_b \\ p_a^4 \end{Bmatrix} = \frac{1}{\sqrt{N}}\left(\frac{2\pi m}{\beta}\right)^{(N-1)/2} \times \begin{Bmatrix} 1 \\ \left(\frac{m}{\beta}\right)\times\begin{cases} \frac{N-1}{N} & a=b \\ -\frac{1}{N} & a\neq b \end{cases} \\ \left(\frac{m}{\beta}\right)^2 \frac{3(N-1)^2}{N^2} \end{Bmatrix}. \tag{51}$$

Hence we obtain

$$\int \frac{dk}{2\pi}\int d\mathbf{p} \exp\left(ik\sum_{a=1}^{N} p_a - \beta\sum_{a=1}^{N}\frac{p_a^2}{2m}\right)\left(1 - \frac{\beta}{c^2} H_R\right)$$
$$= \frac{1}{\sqrt{N}}\left(\frac{2\pi m}{\beta}\right)^{(N-1)/2}\left\{\left(1 - \frac{\beta}{c^2}(\pi G)^2 \sum_{a=1}^{N}\sum_{b=1}^{N}\sum_{c=1}^{N} m^3[|r_{ab}||r_{ac}| - r_{ab}r_{ac}]\right) - \frac{\pi G \beta}{c^2}\left(\frac{m}{\beta}\right) \times \frac{N-1}{N}\sum_{a=1}^{N}\sum_{b=1}^{N}|r_{ab}| + \frac{2\pi G\beta}{c^2}\left(-\frac{m}{N\beta}\right)\sum_{a>b}^{N}|r_{ab}| + \frac{\beta}{8m^3c^2}\frac{3(N-1)^2}{N^2}\left(\frac{m}{\beta}\right)^2\left(\sum_{a=1}^{N} 1\right)\right\}$$
$$= \left(1 - \frac{\beta(\pi G)^2}{c^2}\sum_{a=1}^{N}\sum_{b=1}^{N}\sum_{c=1}^{N} m^3[|r_{ab}||r_{ac}| - r_{ab}r_{ac}] - \frac{2\pi G m}{c^2}\sum_{a>b}^{N}|r_{ab}| + \frac{3(N-1)^2}{8N\beta m c^2}\right)\frac{1}{\sqrt{N}}\left(\frac{2\pi m}{\beta}\right)^{(N-1)/2}. \tag{52}$$

We next consider the integration over the spatial variables. Introducing

$$u_l = z_{l+1} - z_l \quad 1 \leqslant l \leqslant N, \quad u_N = \frac{1}{N}\sum_{m=1}^{N} z_m, \tag{53}$$

we have

$$\sum_{a>b}^{N} |r_{ab}| = \frac{1}{2}\sum_{a=1}^{N}\sum_{b=1}^{N}|r_{ab}| = \sum_{l=1}^{N-1} l(N-l)(z_{l+1}-z_l) = \sum_{l=1}^{N-1} l(N-l)u_l, \tag{54}$$

where without loss of generality the particles are ordered in the sequence $z_1 \leqslant z_2 \leqslant \cdots \leqslant z_N$, and the overall result is then multiplied by **N**! This gives

$$\begin{aligned}\int d\mathbf{z}\,\delta(\overline{\mathbf{z}})F(\mathbf{z}) &= \mathbf{N}!\int_{-\infty}^{\infty}dz_1\int_{z_1}^{\infty}dz_2\int_{z_2}^{\infty}dz_3\cdots\int_{z_{N-1}}^{\infty}dz_N\,\delta(\overline{z})\mathsf{F}(\mathbf{z})\\ &= \mathbf{N}!\int_{-\infty}^{\infty}du_N\int_0^{\infty}du_1\int_0^{\infty}du_2\cdots\int_0^{\infty}du_{N\dots1}\,\delta(u_N)\mathsf{F}(\mathbf{u})\\ &= \mathbf{N}!\int d\mathbf{u}\,\mathsf{F}(\mathbf{u}),\end{aligned} \tag{55}$$

provided the function $\mathsf{F}(\mathbf{z})$ is symmetric under interchange of any pair of variables, which is the case here. The inverse transformation reads

$$z_n = u_N - \frac{1}{N}\sum_{l=n}^{N-1} D_{n,l} \quad \text{where} \quad D_{n,l} = \begin{cases} -l, & n>l\\ N-l, & n\leqslant l,\end{cases} \tag{56}$$

and so we have

$$\begin{aligned}\sum_{a=1}^{N}\sum_{b=1}^{N}\sum_{c=1}^{N}[|r_{ab}||r_{ac}| - r_{ab}r_{ac}] &= 2\left(\sum_{b>a>c}^{N} r_{ba}r_{ac} + \sum_{b>a>c}^{N} r_{ca}r_{ab}\right)\\ &= 4\sum_{k=1}^{N-2}\sum_{l=k+1}^{N-1}(N-l)(l-k)ku_lu_k.\end{aligned} \tag{57}$$

Consequently the partition function is

$$\begin{aligned}\mathcal{Z} = \frac{e^{-\beta Mc^2}}{\sqrt{N}}\left(\frac{2\pi m}{\beta}\right)^{(N-1)/2}\int_0^{\infty}du_1\int_0^{\infty}du_2\cdots\int_0^{\infty}du_{N-1}\exp\left(-2\pi G\beta m^2\sum_{n=1}^{N-1}n(N-n)u_n\right)\\ \times\left(1 - \frac{4\beta m^3}{c^2}(\pi G)^2\sum_{k=1}^{N-2}\sum_{l=k+1}^{N-1}(N-l)(l-k)ku_lu_k - \frac{2\pi Gm}{c^2}\sum_{l=1}^{N-1}l(N-l)u_l + \frac{3(N-1)^2}{8N\beta mc^2}\right).\end{aligned} \tag{58}$$

The three basic integrals in Eq. (58) are

$$\int d\mathbf{u}\exp\left(-\lambda\sum_{n=1}^{N-1}n(N-n)u_n\right) = \frac{1}{\lambda^{N-1}[(N-1)!]^2}, \tag{59}$$

$$\int d\mathbf{u}\sum_{l=1}^{N-1}k(N-k)u_k\exp\left(-\lambda\sum_{n=1}^{N-1}n(N-n)u_n\right) = \frac{N-1}{\lambda^{N}[(N-1)!]^2}, \tag{60}$$

$$\int d\mathbf{u}\sum_{k=1}^{N-2}\sum_{l=k+1}^{N-1}(N-l)(l-k)ku_lu_k\exp\left(-\lambda\sum_{n=1}^{N-1}n(N-n)u_n\right) = \frac{\displaystyle\sum_{k=1}^{N-2}\sum_{l=k+1}^{N-1}\frac{(l-k)}{l(N-k)}}{\lambda^{N+1}[(N-1)!]^2}, \tag{61}$$

yielding

$$
\mathcal{Z}=e^{-\beta Mc^2}\frac{1}{\sqrt{N}}\left(\frac{2\pi m}{\beta}\right)^{(N-1)/2}\left(\frac{1+\dfrac{3(N-1)^2}{8N\beta mc^2}}{(2\pi G\beta m^2)^{N-1}[(N-1)!]^2}-\frac{4\beta}{c^2}\frac{(\pi G)^2m^3}{(2\pi G\beta m^2)^{N+1}[(N-1)!]^2}\right.
$$

$$
\left.\times\sum_{k=1}^{N-1}\sum_{l=k+1}^{N-1}\frac{(l-k)}{l(N-k)}-\frac{2\pi Gm}{c^2}\frac{N-1}{(2\pi G\beta m^2)^N[(N-1)!]^2}\right)
$$

$$
=\frac{e^{-\beta Mc^2}\left(\dfrac{2\pi m}{\beta}\right)^{(N-1)/2}}{\sqrt{N}(2\pi G\beta m^2)^{N-1}[(N-1)!]^2}\left[1-\frac{1}{\beta mc^2}\left\{\frac{(5N+3)(N-1)+8N\sum_{k=1}^{N-1}\sum_{l=k+1}^{N-1}\dfrac{(l-k)}{l(N-k)}}{8N}\right\}\right]
$$

$$
=\frac{\exp\left[-\beta Mc^2-\dfrac{3(N-1)}{2}\ln(\beta mc^2)-\left\{\dfrac{(5N+3)(N-1)+8N\sum_{k=1}^{N-1}\sum_{l=k+1}^{N-1}\dfrac{(l-k)}{(N-k)}}{8N\beta mc^2}\right\}\right]}{\sqrt{N}(\sqrt{2\pi}G/c^3)^{(N-1)}[(N-1)!]^2}, \tag{62}
$$

which is the partition function to lowest relativistic order. The average energy is

$$
\langle E\rangle=-\frac{\partial}{\partial\beta}\ln\mathcal{Z}=Mc^2+\frac{3}{2}(N-1)\beta^{-1}-\left\{\frac{(5N+3)(N-1)+8N\sum_{k=1}^{N-1}\sum_{l=k+1}^{N-1}\dfrac{(l-k)}{l(N-k)}}{8Mc^2}\right\}\beta^{-2}, \tag{63}
$$

to the relevant order in c^{-2}. The relativistic correction grows quadratically with N (for fixed $M=Nm$) and is negative. Hence the average energy of a relativistic gravitating system is lower than its nonrelativistic counterpart at the same temperature.

B. The single-particle distribution function

Consider next the one-particle distribution function, which is

$$
f_c^R(p,z)=\frac{1}{N}\sum_{n=1}^{N}f_{cn}^R(p,z), \tag{64}
$$

where

$$
f_{cn}^R(p,z)=\frac{e^{-\beta Mc^2}}{\mathcal{Z}N!}\int d\mathbf{z}\,\delta(\overline{z})\exp\left(-2\pi G\beta m^2\sum_{a>b}^{N}|r_{ab}|\right)\delta(z-z_n)\int\frac{dk}{2\pi}\int d\mathbf{p}\exp\left(ik\sum_{a=1}^{N}p_a-\beta\sum_{a=1}^{N}\frac{p_a^2}{2m}\right)
$$

$$
\times\left(1-\frac{\beta}{c^2}H_R\right)\delta(p-p_n)
$$

$$
=\frac{c^{-\beta Mc^2}}{\mathcal{Z}}\int_{-\infty}^{\infty}du_N\int_0^{\infty}du_1\int_0^{\infty}du_2\cdots\int_0^{\infty}du_{N-1}\,\delta(u_N)\exp\left(-2\pi G\beta m^2\sum_{l=1}^{N-1}l(N-l)u_l\right)
$$

$$
\times\delta\left(z-u_N+\frac{1}{N}\sum_{l=1}^{N-1}D_{n,l}u_l\right)\theta_{cn}(p,\mathbf{z})
$$

$$
=\frac{e^{-\beta Mc^2}}{\mathcal{Z}}\int_0^{\infty}du_1\cdots\int_0^{\infty}du_{N-1}\exp\left(-2\pi G\beta m^2\sum_{l=1}^{N-1}C_lu_l\right)\delta\left(z+\frac{1}{N}\sum_{l=1}^{N-1}D_{n,l}u_l\right)\theta_{cn}(p,\mathbf{z}), \tag{65}
$$

in which

$$\theta_{cn}(p,\mathbf{z}) = \int \frac{dk}{2\pi} \int d\mathbf{p} \exp\left(ik\sum_{a=1}^{N} p_a - \beta \sum_{a=1}^{N} \frac{p_a^2}{2m} \right) \left(1 - \frac{\beta}{c^2} H_R \right) \delta(p - p_n), \tag{66}$$

and

$$C_l = l(N-l), \quad D_{n,l} = \begin{cases} (N-l), & n \leqslant l \\ -l, & n > l, \end{cases} \tag{67}$$

where Eq. (53) has been used to express

$$z_n = u_N + \frac{1}{N} \sum_{l=1}^{N-1} l u_l - \sum_{l=n}^{N-1} u_l = u_N - \frac{1}{N} \sum_{l=1}^{N-1} D_{n,l} u_l . \tag{68}$$

Evaluation of $\theta_{cn}(p,\mathbf{z})$ is somewhat lengthy, and so we relegate its computation to the Appendix. We obtain

$$\begin{aligned} \theta_{cn}(p,\mathbf{z}) = {} & \frac{1}{\sqrt{N-1}} \exp\left(-\frac{N\beta p^2}{2m(N-1)} \right) \left(\frac{2\pi m}{\beta} \right)^{(N-2)/2} \Bigg\{ \Bigg(1 - \frac{\beta m (\pi G m)^2}{c^2} \\ & \times \sum_{n=1}^{N} \sum_{b=1}^{N} \sum_{c=1}^{N} [|r_{ab}||r_{ac}| - r_{ab} r_{ac}] \Bigg) + \frac{1}{2\beta m c^2} \left[\frac{\beta^2 p^4 N (N^2 - 3N + 3)}{(2m)^2 (N-1)^3} + \frac{3\beta p^2 (N-2)}{2m(N-1)^2} + \frac{3(N-2)^2}{4(N-1)} \right] \\ & + \left(\frac{2\pi m G}{c^2} \right) \left[-\frac{1}{2} \sum_{a=1}^{N} \sum_{b=1}^{N} |r_{ab}| - \left(\frac{N^2 \beta p^2}{2m(N-1)^2} - \frac{N}{2(N-1)} \right) \left(\sum_{c=1}^{N} |r_{cn}| \right) \right] \Bigg\}, \end{aligned} \tag{69}$$

or alternatively, in terms of the **u** variables

$$\begin{aligned} \theta_{cn}(p,\mathbf{u}) = {} & \frac{1}{\sqrt{N-1}} \exp\left(-\frac{N\beta p^2}{2m(N-1)} \right) \left(\frac{2\pi m}{\beta} \right)^{(N-2)/2} \Bigg\{ \left(1 - \frac{4\beta m (\pi G m)^2}{c^2} \sum_{k=1}^{N-2} \sum_{l=k+1}^{N-1} (N-l)(l-k) k u_l u_k \right) \\ & + \frac{1}{2\beta m c^2} \left[\frac{\beta^2 p^4 N (N^2 - 3N + 3)}{(2m)^2 (N-1)^3} + \frac{3\beta p^2 (N-2)}{2m(N-1)^2} + \frac{3(N-2)^2}{4(N-1)} \right] \\ & - \left(\frac{2\pi m G}{c^2} \right) \left[\sum_{l=1}^{N-1} l(N-l) u_l + \left(\frac{N^2 \beta p^2}{2m(N-1)^2} - \frac{N}{2(N-1)} \right) \left(\sum_{s=1}^{n-1} s u_s + \sum_{s=n}^{N-1} (N-s) u_s \right) \right] \Bigg\}. \end{aligned} \tag{70}$$

Now consider Eq. (65), which can be rewritten as

$$f_{cn}^{R}(p,z) = \frac{e^{-\beta M c^2}}{\mathcal{Z}} \int \frac{dk}{2\pi} \int d\mathbf{u} \exp\left(-ikz - \lambda\beta \sum_{l=1}^{N-1} (C_l + i\alpha D_{nl}) u_l \right) \theta_{cn}(p,\mathbf{u}), \tag{71}$$

where

$$\lambda = 2\pi G m^2, \quad \alpha = \frac{k}{N\beta\lambda}. \tag{72}$$

The integration now involves straightforward integrations over the **u** variables, after which an evaluation of the k integral using Jordan's lemma must be performed. This involves some rather tedious manipulations which we describe in the Appendix. The final result is

$$f_{cn}(p,z)=\frac{(2\pi Gm^2)(N\beta)^{3/2}}{\sqrt{2\pi m(N-1)}}\exp\left[\frac{1}{\beta mc^2}\left\{\frac{(5N+3)(N-1)}{8N}+\sum_{k=1}^{N-1}\sum_{l=k+1}^{N-1}\frac{(l-k)}{l(N-k)}\right\}\right]$$
$$\times\sum_{l=1}^{N-1}\left[\left\{A_l^N\left[1+\frac{1}{2\beta mc^2}\left(\frac{\beta^2p^4[1+(N-1)^3]}{(2m)^2(N-1)^3}+\frac{3(N-2)\beta p^2}{2m(N-1)^2}+\frac{3(N-2)^2}{4(N-1)}\right)\right]\right.\right.$$
$$-\frac{1}{\beta mc^2}\{B_l^N-A_l^N[1-2N(\beta\pi Gm^2)l|z|]\}+\frac{2}{\beta mc^2}\left(\frac{N}{2(N-1)}-\frac{\beta p^2}{2m}\left[\frac{N^2}{(N-1)^2}\right]\right)$$
$$\left.\times\left[C_l^N-\frac{1}{l}A_l^N[1-2N(\beta\pi Gm^2)l|z|]\right]-\frac{1}{\beta mc^2}\{D_l^N+K_l^N[1-2N(\beta\pi Gm^2)l|z|]\}\right\}$$
$$\left.\times\exp\left(-\frac{N\beta p^2}{2m(N-1)}-2\pi GN\beta m^2l|z|\right)\right], \tag{73}$$

where A_l^N, B_l^N, C_l^N, D_l^N, and K_l^N are defined in Sec. 4 of the Appendix. Integration over p yields the canonical density distribution function

$$\rho_c(z)=\int_{-\infty}^{\infty}dp\, f_{cn}(p,z)$$
$$=(2\pi Gm^2N\beta)\exp\left[\frac{1}{\beta mc^2}\left\{\frac{(5N+3)(N-1)}{8N}\right.\right.$$
$$\left.\left.+\sum_{k=1}^{N-1}\sum_{l=k+1}^{N-1}\frac{(l-k)}{l(N-k)}\right\}\right]$$
$$\times\sum_{l=1}^{N-1}\left\{A_l^N+\frac{1}{\beta mc^2}\left(\frac{3}{8}\frac{(N-1)^2}{N}A_l^N-B_l^N-D_l^N\right)\right.$$
$$\left.+\frac{1}{\beta mc^2}[A_l^N-K_l^N](1-2\pi Gm^2N\beta l|z|)\right\}$$
$$\times\exp(-2\pi Gm^2N\beta l|z|), \tag{74}$$

whereas integration over z yields the canonical momentum distribution function

$$\vartheta_{cn}(p)=\int_{-\infty}^{\infty}dz\, f_{cn}(p,z)$$
$$=\sqrt{\frac{(N\beta)}{2\pi m(N-1)}}\exp\left[-\frac{N\beta p^2}{2m(N-1)}\right]$$
$$\times\left[1+\frac{1}{\beta mc^2}\left(\frac{N\beta^2p^4(N^2-3N+3)}{8m^2(N-1)^3}\right.\right.$$
$$\left.\left.-\frac{\beta p^2(4N^2-7N+6)}{4m(N-1)^2}+\frac{5N(N-1)+3}{8N(N-1)}\right)\right]. \tag{75}$$

From the results in the Appendix, it can be shown that

$$\sum_{l=1}^{N-1}\frac{1}{l}A_l^N=\frac{1}{2},\quad \sum_{l=1}^{N-1}\frac{1}{l}B_l^N=\frac{1}{2}(N-1),$$
$$\sum_{l=1}^{N-1}\frac{1}{l}C_l^N=\frac{(N-1)}{N},$$
$$\sum_{l=1}^{N-1}\frac{1}{l}D_l^N=\frac{1}{2}\sum_{k=1}^{N-1}\sum_{l=k+1}^{N-1}\frac{(l-k)}{l(N-k)}, \tag{76}$$

which can be used to show that

$$\int_{-\infty}^{\infty}dz\,\rho_c(z)=1\quad\text{and}\quad\int_{-\infty}^{\infty}dp\,\vartheta_{cn}(p)=1. \tag{77}$$

We also have the relations

$$\sum_{l=1}^{N-1}A_l^N=\frac{1}{2}\frac{N-1}{2N-3},\quad \sum_{l=1}^{N-1}B_l^N=\frac{1}{2}\frac{(N-1)^2}{2N-3}, \tag{78}$$

the first of which was demonstrated by Rybicki [1].

V. THE MICROCANONICAL ENSEMBLE

The results for the canonical ensemble obtained in the previous section are for a system of relativistic gravitating particles coupled to a heat bath that keeps the system at a constant temperature $T=\beta^{-1}$. In such a situation the energy of the system is ill defined, and undergoes fluctuations of the order kT. An isolated system, on the other hand, would have its total energy conserved, and this is the more realistic astrophysical case. This entails usage of the microcanonical ensemble, in which phase space integrations are carried out by constraining the total energy to be E. The weighting function $e^{-\beta H}$ in the phase space integral is, therefore, replaced with $\delta(E-H)$.

Fortunately it is straightforward to compute the relevant microcanonical quantities from the canonical ones. Using the same reasoning that led to Eq. (42), the microcanonical single-particle distribution function is

$$f'^{R}_{mc}(p,z)=\frac{1}{\Omega N!}\int\int d\mathbf{p}\,d\mathbf{z}\,\delta(\overline{p})\,\delta(\overline{z})\,\delta(E-H)N^{-1}\times\sum_{a}\delta(z-z_a)\,\delta(p-p_a), \qquad (79)$$

where

$$\Omega=\frac{1}{N!}\int\int d\mathbf{p}\,d\mathbf{z}\,\delta(\overline{p})\,\delta(\overline{z})\,\delta(E-H). \qquad (80)$$

Note that Ω and $\mathcal{Z}$ are related by the Laplace transforms

$$\mathcal{Z}=\int_0^\infty dE\,e^{-\beta H}\Omega(E), \qquad (81)$$

$$\Omega=\frac{1}{2\pi i}\int_C d\beta\,e^{\beta E}\mathcal{Z}(\beta), \qquad (82)$$

where the contour $\mathcal{C}$ in the latter integral extends from $-i\infty$ to $+i\infty$ to the right of all singularities. Using the general result that

$$\frac{(w)_+^{\zeta-1}}{\Gamma(\zeta)}=\frac{1}{2\pi i}\int_C d\beta\,e^{\beta w}\beta^{-\zeta}\quad\text{where}\quad (w)_+=\begin{cases}w, & w\geqslant 0\\ 0, & w<0,\end{cases} \qquad (83)$$

it is straightforward to obtain

$$\Omega=\frac{(2\pi m)^{(N-1)/2}(E-Mc^2)^{(3N-5)/2}}{\sqrt{N}(\pi G m^2)^{N-1}[(N-1)!]^2\Gamma\left(\frac{3}{2}(N-1)\right)}\exp\left[-\frac{(E-Mc^2)}{Mc^2}\left\{\frac{(5N+3)(N-1)+8N\sum_{k=1}^{N-1}\sum_{l=k+1}^{N-1}\frac{(l-k)}{l(N-k)}}{12(N-1)}\right\}\right], \qquad (84)$$

to leading order in $1/c$.

Similarly using Eq. (83) in Eq. (79), we have

$$\Omega f'^{R}_{mc}(p,z)=\frac{1}{2\pi i}\int_C d\beta\,e^{\beta\varepsilon}[\mathcal{Z}f^{R}_{cn}(p,z)]. \qquad (85)$$

Using Eqs. (62) and (73) we have

$$\begin{aligned}f_{cn}(p,z)=&\frac{(2\pi G m^2)(N\beta)^{3/2}}{\sqrt{2\pi m(N-1)}}\exp\left[\frac{1}{\beta mc^2}\left\{\frac{(5N+3)(N-1)}{8N}+\sum_{k=1}^{N-1}\sum_{l=k+1}^{N-1}\frac{(l-k)}{l(N-k)}\right\}\right]\\&\times\sum_{l=1}^{N-1}\left[\left\{A_l^N\left(1+\frac{1}{2\beta mc^2}\left(\frac{\beta^2p^4[1+(N-1)^3]}{(2m)^2(N-1)^3}+\frac{3(N-2)\beta p^2}{2m(N-1)^2}+\frac{3(N-2)^2}{4(N-1)}\right)\right)\right.\right.\\&-\frac{1}{\beta mc^2}\{B_l^N-A_l^N[1-2N(\beta\pi G m^2)l|z|]\}+\frac{2}{\beta mc^2}\left(\frac{N}{2(N-1)}-\frac{\beta p^2}{2m}\left[\frac{N^2}{(N-1)^2}\right]\right)\\&\left.\times\left[C_l^N-\frac{1}{l}A_l^N[1-2N(\beta\pi G m^2)l|z|]\right]-\frac{1}{\beta mc^2}\{D_l^N+K_l^N[1-2N(\beta\pi G m^2)l|z|]\}\right\}\\&\left.\times\exp\left(\frac{N\beta p^2}{2m(N-1)}-2\pi G N\beta m^2 l|z|\right)\right], \end{aligned} \qquad (86)$$

$$f'^{R}_{mc}(p,z)=\frac{2\pi Gm^2}{\sqrt{2\pi m(N-1)}}\left(\frac{N}{(E-Mc^2)}\right)^{3/2}\frac{\Gamma\left(\frac{3}{2}(N-1)\right)}{\Gamma\left(\frac{3}{2}(N-2)\right)}\exp\left[\frac{(E-Mc^2)}{Mc^2}\left\{\frac{(5N+3)(N-1)+8N\sum_{k=1}^{N-1}\sum_{j=k+1}^{N-1}\frac{(j-k)}{j(N-k)}}{12(N-1)}\right\}\right]$$
$$\times\sum_{l=1}^{N-1}\left\{\left[A_l^N+\left(\frac{A_l^N}{mc^2}\left[\frac{3p^2(N-2)}{4m(N-1)^2}\right]-\frac{C_l^N-\frac{1}{l}A_l^N}{mc^2}\left[\frac{2N^2p^2}{2m(N-1)^2}\right]\right)\right]\Upsilon_+^{3N/2-4}\right.$$
$$+\left\{\frac{2N(\pi Gm^2)|z|}{mc^2}\left[A_l^N\left(\frac{N}{(N-1)}-l\right)+lK_l^N\right]\right\}\Upsilon_+^{3N/2-4}$$
$$+\frac{\left(\frac{3}{2}N-4\right)}{(E-Mc^2)}\frac{A_l^N}{2mc^2}\left[\frac{p^4N(N^2-3N+3)}{(2m)^2(N-1)^3}-\frac{2p^2}{m}\left(\frac{2N^3\pi Gm^2|z|}{(N-1)^2}\right)\right]\Upsilon_+^{3N/2-5}$$
$$\left.+\frac{(E-Mc^2)}{\left(\frac{3}{2}N-3\right)mc^2}\left(A_l^N\left[\frac{3(N-2)^2}{8(N-1)}+1\right]-B_l^N+\frac{N(C_l^N-lA_l^N)}{(N-1)}-D_l^N-K_l^N\right)\Upsilon_+^{3N/2-3}\right\},\qquad(87)$$

as the expression for the relativistic microcanonical partition function, valid to $O(1/c^2)$, where

$$\Upsilon(p,z)\equiv1-\frac{Np^2}{2m(N-1)(E-Mc^2)}-\frac{2N\pi Gm^2}{(E-Mc^2)}l|z|.\qquad(88)$$

Employing the expressions

$$\int_{-1}^{1}dy(1-y^2)^{\gamma}=\sqrt{\pi}\frac{\Gamma(\gamma+1)}{\Gamma\left(\gamma+\frac{3}{2}\right)},\qquad(89)$$

$$\int_{-1}^{1}dy\,y^2(1-y^2)^{\gamma}=\sqrt{\pi}\frac{\Gamma(\gamma+1)}{2\Gamma\left(\gamma+\frac{5}{2}\right)},\qquad(90)$$

$$\int_{-1}^{1}dy\,y^4(1-y^2)^{\gamma}=\sqrt{\pi}\frac{3\Gamma(\gamma+1)}{4\Gamma\left(\gamma+\frac{7}{2}\right)},\qquad(91)$$

the density distribution is

$$\rho_{mc}(z)=\int_{-\infty}^{\infty}dp\,f'^{R}_{mc}(p,z)$$
$$=\left(\frac{2N\pi Gm^2}{(E-Mc^2)}\right)\exp\left[\frac{(E-Mc^2)}{Mc^2}\left\{\frac{(5N+3)(N-1)+8N\sum_{k=1}^{N-1}\sum_{j=k+1}^{N-1}\frac{(j-k)}{j(N-k)}}{12(N-1)}\right\}\right]$$
$$\times\sum_{l=1}^{N-1}\left\{\left(\frac{3N-5}{2}\right)\left[A_l^N+\frac{2(\pi GM^2)|z|}{Mc^2}(-lA_l^N+lK_l^N)\right][\Upsilon(0,z)]_+^{3N/2-7/2}\right.$$

$$+\left(\frac{(E-Mc^2)}{Mc^2}\right)[\Upsilon(0,z)]_+^{3N/2-5/2}\left[\left[\left\{-\left(C_l^N-\frac{1}{l}A_l^N\right)\left[\frac{N^2}{(N-1)}\right]\right\}+A_l^N\left[\frac{3(N^2-3N+3)}{8(N-1)}+\frac{3(N-2)}{4(N-1)}+\frac{3N(N-2)^2}{8(N-1)}+N\right]\right.\right.$$

$$\left.\left.+\left(-NB_l^N+\frac{N^2\left(C_l^N-\frac{1}{l}A_l^N\right)}{(N-1)}-N(D_l^N+K_l^N)\right)\right]\right\}$$

$$=\left(\frac{2N\pi Gm^2}{(E-Mc^2)}\right)\exp\left[\frac{(E-Mc^2)}{Mc^2}\left\{\frac{(5N+3)(N-1)+8N\sum_{k=1}^{N-1}\sum_{j=k+1}^{N-1}\frac{(j-k)}{j(N-k)}}{12(N-1)}\right\}\right]$$

$$\times\sum_{l=1}^{N-1}\left\{\left(\frac{3N-5}{2}\right)\left[A_l^N-\frac{2(\pi CM^2)l|z|}{Mc^2}(A_l^N-K_l^N)\right][\Upsilon(0,z)]_+^{3N/2-7/2}\right.$$

$$\left.+\left(\frac{(E-Mc^2)}{Mc^2}\right)[\Upsilon(0,z)]_+^{3N/2-5/2}\left[\left(\frac{3(N-1)^2}{8}\right)A_l^N-N(B_l^N-A_l^N)-N(D_l^N+K_l^N)\right]\right\}. \tag{92}$$

The normalization of the density $\int_{-\infty}^{\infty}dz\,\rho_{mc}(z)=1$ implies

$$2\sum_{l=1}^{N-1}\left\{\frac{A_l^N}{l}+\left(\frac{(E-Mc^2)}{Mc^2}\right)\left[\frac{A_l^N}{l}\left(\frac{(N-1)}{4}\right)-\frac{2NB_l^N}{3(N-1)l}-\frac{2N}{3(N-1)}\frac{D_l^N}{l}\right]\right\}$$

$$=\exp\left[-\frac{(E-Mc^2)}{Mc^2}\left\{\frac{(5N+3)(N-1)+8N\sum_{k=1}^{N-1}\sum_{j=k+1}^{N-1}\frac{(j-k)}{j(N-k)}}{12(N-1)}\right\}\right], \tag{93}$$

which, using Eq. (76), is easily shown to be satisfied to first order in $\zeta\equiv(E-Mc^2)/Mc^2$, where the latter quantity is the dimensionless fraction of excess energy above the total rest mass.

The momentum distribution is

$$\vartheta_{mc}(p)=\int_{-\infty}^{\infty}dz\,f'^{R}_{mc}(p,z)=\left(\frac{N}{2\pi m(N-1)(E-Mc^2)}\right)^{1/2}\frac{\Gamma\left(\frac{3}{2}(N-1)\right)}{\Gamma\left(\frac{3}{2}N-2\right)}$$

$$\times\exp\left[\frac{(E-Mc^2)}{Mc^2}\left\{\frac{(5N+3)(N-1)+8N\sum_{k=1}^{N-1}\sum_{j=k+1}^{N-1}\frac{(j-k)}{j(N-k)}}{12(N-1)}\right\}\right]$$

$$\times\sum_{l=1}^{N-1}\left\{\left[\frac{2A_l^N}{l}+\left(\frac{2A_l^N}{lmc^2}\left[\frac{3p^2(N-2)}{4m(N-1)^2}\right]-\frac{2\left(C_l^N-\frac{1}{l}A_l^N\right)}{lmc^2}\left[\frac{N^2p^2}{m(N-1)^2}\right]\right)\right][\Upsilon(p,0)]_+^{3N/2-3}\right.$$

$$+\left\{\frac{4(E-Mc^2)}{mc^2(3N-4)}\left[\frac{A_l^N}{l^2}\left(\frac{N}{(N-1)}-l\right)+\frac{K_l^N}{l}\right]\right\}[\Upsilon(p,0)]_+^{3N/2-2}+\frac{2A_l^N}{l^2mc^2}\left[-\frac{p^2}{m}\left(\frac{N^2}{(N-1)^2}\right)\right][\Upsilon(p,0)]_+^{3N/2-3}$$

$$+\frac{\left(\frac{3}{2}N-3\right)}{(E-Mc^2)}\frac{A_l^N}{lmc^2}\left[\frac{p^4N(N^2-3N+3)}{(2m)^2(N-1)^3}\right][\Upsilon(p,0)]_+^{3N/2-4}+\frac{4(E-Mc^2)}{(3M-4)mc^2}$$

$$\left.\times\left(\frac{A_l^N}{l}\left[\frac{3(N-2)^2}{8(N-1)}+1\right]-\frac{B_l^N}{l}+\frac{N\left(C_l^N-\frac{1}{l}A_l^N\right)}{l(N-1)}-\frac{D_l^N+K_l^N}{l}\right)[\Upsilon(p,0)]_+^{3N/2-2}\right\}$$

$$=\left(\frac{N}{2\pi m(N-1)(E-Mc^2)}\right)^{1/2}\frac{\Gamma\left(\frac{3}{2}(N-1)\right)}{\Gamma\left(\frac{3}{2}N-2\right)}\exp\left[\frac{(E-Mc^2)}{Mc^2}\left\{\frac{(5N+3)(N-1)+8N\sum_{k=1}^{N-1}\sum_{j=k+1}^{N-1}\frac{(j-k)}{j(N-k)}}{12(N-1)}\right\}\right]$$

$$\times\left\{\left[1-\frac{N}{Mc^2}\left(\frac{p^2}{4m}\left[\frac{8N^2-11N+6}{(N-1)^2}\right]\right)\right][\Upsilon(p,0)]_+^{3N/2-3}+\frac{N}{(E-Mc^2)}\frac{3(N-2)}{4Mc^2}\left[\frac{p^4N(N^2-3N+3)}{(2m)^2(N-1)^3}\right][\Upsilon(p,0)]_+^{3N/2-4}\right.$$

$$\left.-\frac{2N(E-Mc^2)}{(3N-4)Mc^2}\left(\frac{(5N^2-20N+12)}{8(N-1)}+\sum_{s=1}^{N-1}\sum_{t=s+1}^{N-1}\frac{(t-s)}{t(N-s)}\right)[\Upsilon(p,0)]_+^{3N/2-2}\right\}. \tag{94}$$

It is straightforward to show that $\int_{-\infty}^{\infty}dp\;\vartheta_{mc}(p)=1$ from Eqs. (89)–(91) to first order in $\zeta\equiv(E-Mc^2)/Mc^2$.

VI. THE LARGE-N LIMIT

For statistical systems (such as those of interest in stellar dynamics), the large-N limit is of considerable physical interest. This is the limit in which the total energy E and total mass $M=Nm$ are fixed. In the nonrelativistic case, the single-particle distribution function approaches the isothermal solution of the Vlasov equations in the large-N limit [1]. However, the relativistic case is somewhat more subtle, since the expressions we have obtained are valid only if the speed of light is sufficiently large relative to other quantities of the same dimension, and as N becomes large we must ensure that this approximation remains valid.

In order to investigate the large-N limit it is necessary to rewrite all quantities in terms of E, M, and N. As in Ref. [1], we adopt the dimensionless variables

$$\eta\equiv\frac{p}{mV},\quad \xi\equiv\frac{z}{L}, \tag{95}$$

where

$$L\equiv\frac{2(E-Mc^2)}{3\pi GM^2}=\frac{2\zeta c^2}{3\pi GM},\quad V^2\equiv\frac{4(E-Mc^2)}{3M}=\frac{4\zeta c^2}{3} \tag{96}$$

are the characteristic length and velocity scales of the system. The scaled distributions functions are correspondingly defined

$$\rho^*(\xi)\equiv L\rho(L\xi),\quad \vartheta^*(\eta)\equiv mV\vartheta(mV\eta),\quad f'^{*R}(\eta,\xi)\equiv mVLf'^R(mV\eta,L\xi), \tag{97}$$

so that

$$\int\int d\eta\,d\xi\,f'^{*R}(\eta,\xi)=\int d\xi\,\rho^*(\xi)=\int d\eta\,\vartheta^*(\eta)=1. \tag{98}$$

Consider first the partition function (62), which can be rewritten as

$$\mathcal{Z}=\frac{\exp\left[-\beta Mc^2-\frac{1}{\beta Mc^2}\left\{\frac{(5N+3)(N-1)+8N\sum_{k=1}^{N-1}\sum_{l=k+1}^{N-1}\frac{(l-k)}{l(N-k)}}{8}\right\}\right]}{\sqrt{N}(\sqrt{2\pi}G/c^3)^{(N-1)}[(N-1)!]^2}\left(\frac{N}{\beta Mc^2}\right)^{3(N-1)/2}. \tag{99}$$

The approximation (47) is valid provided

$$\beta>\frac{(5N+3)(N-1)+8N\sum_{k=1}^{N-1}\sum_{l=k+1}^{N-1}\frac{(l-k)}{l(N-k)}}{8Mc^2}, \tag{100}$$

which sets an upper bound on the thermal energy $kT=\beta^{-1}$ of the system for a given value of N. For fixed Mc^2 the relativistic corrections are valid only as the temperature becomes vanishingly small in the limit of large N. The exponential approximation is slightly better than the polynomial one because of the positivity of the partition function.

Consider next the average energy in the canonical case as

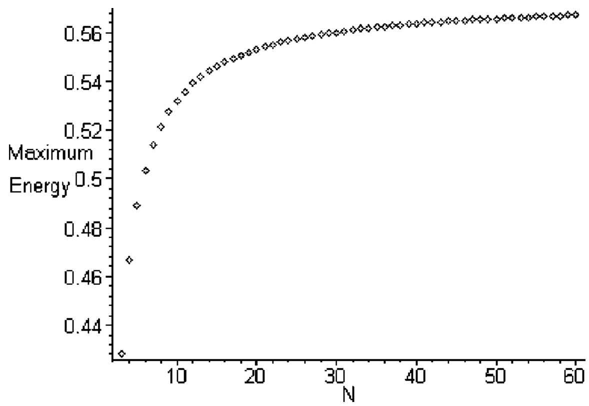

FIG. 1. Maximum value of the average relativistic energy in units of Mc^2 as a function of *N*.

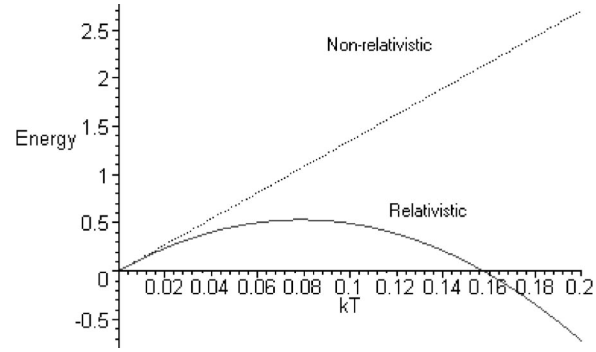

FIG. 2. Average energy $\langle\zeta\rangle$ as a function of kT for $N=10$ for the nonrelativistic and relativistic cases. Axes are in units of Mc^2.

given by Eq. (63), which we rewrite as

$$\langle\zeta\rangle=\frac{3}{2}(N-1)\frac{1}{\beta Mc^2}-\left\{\frac{(5N+3)(N-1)+8N\sum_{k=1}^{N-1}\sum_{l=k+1}^{N-1}\frac{(l-k)}{l(N-k)}}{8}\right\}\times\left(\frac{1}{\beta Mc^2}\right)^2, \tag{101}$$

where $\zeta\equiv(E-Mc^2)/Mc^2$, as before. When the thermal energy $kT=\beta^{-1}$ of the system is sufficiently small relative to its rest energy Mc^2, the expression for the average energy $\langle\zeta\rangle$ does not differ much from the nonrelativistic value given by its first term. As the thermal energy grows (i.e., as β decreases) the value of $\langle\zeta\rangle$ increases more slowly than its nonrelativistic counterpart, reaching a maximum when

$$\beta=\frac{(5N+3)(N-1)+8N\sum_{k=1}^{N-1}\sum_{l=k+1}^{N-1}\frac{(l-k)}{l(N-k)}}{6(N-1)Mc^2}\equiv\beta_m, \tag{102}$$

after which the average energy decreases with decreasing β, becoming negative when $\beta=\frac{1}{2}\beta_m$. As N becomes large, $\beta_m\rightarrow[\frac{7}{2}-(2\pi^2/9)](N/Mc^2)$. At $\beta=\beta_m$ the average energy has half the value of its nonrelativistic counterpart. In Fig. 1 we plot the maximum value of $\langle\zeta\rangle$ as a function of *N*. The curve asymptotes to the constant value of $\langle\zeta\rangle=0.573\,940\,872$ as $N\rightarrow\infty$.

Of course the relativistic expansion (44) breaks down well before β reaches this point. In Fig. 2 we plot the average energy $\langle\zeta\rangle$ as a function of kT for $N=10$. The relativistic case is clearly distinguishable from its nonrelativistic counterpart once $kT/Mc^2>0.02$. However, the upper bound on the thermal energy is $kT/Mc^2<0.0117$ for $N=10$. In Fig. 3 we plot the average energy over the allowed range of kT illustrating that the distinction between the two cases is about 8% at most. For $N=1000$ the maximum difference between the two cases is less than one part in a thousand over the allowed range of kT.

In the canonical case we take the energy E to be the fixed total average energy as given by Eq. (63). Solving this equation for the inverse temperature β yields

$$\beta=\frac{18(N-1)-\left[(5N+3)+\frac{8N}{(N-1)}\sum_{k=1}^{N-1}\sum_{l=k+1}^{N-1}\frac{(l-k)}{l(N-l)}\right]\zeta}{12(E-Mc^2)}=\frac{a_N}{Mc^2\zeta}\left(1-\frac{b_N}{a_N^2}\zeta\right), \tag{103}$$

where

$$a_N=\frac{3}{2}(N-1),$$

$$b_N=\frac{1}{8}\left((5N+3)(N-1)+8N\sum_{k=1}^{N-1}\sum_{j=k+1}^{N-1}\frac{(j-k)}{j(N-k)}\right), \tag{104}$$

are defined for convenience. The limit (100) implies that

$$1>\left(\frac{b_N}{a_N^2}+\frac{b_N}{a_N}\right)\zeta\equiv\frac{\zeta}{\zeta_{\max}}\to\left(\frac{7}{4}-\frac{\pi^2}{9}\right)N\zeta, \tag{105}$$

where the latter limit holds for large N. Since the exponential forms a better approximation than the polynomial one, the value of $\zeta_{\max}$ is probably a bit larger than what is given in Eq. (105), although it is not clear how much. We obtain

$$\begin{aligned}
f_c^*(\eta,\xi)&=\frac{M}{N}VLf'^R(mV\eta,L\xi)=\frac{2}{3N\pi G}\sqrt{\frac{4}{3}}(\zeta c^2)^{3/2}f'^R(mV\eta,L\xi)=\frac{2\left(\frac{2}{3}a_N\right)^{3/2}}{N\sqrt{\pi(N-1)}}\\
&\times\left(1-\frac{b_N}{a_N^2}\zeta\right)^{3/2}\exp\left[\frac{b_N}{a_N}\zeta\left(1-\frac{b_N}{a_N^2}\zeta\right)^{-1}\right]\sum_{l=1}^{N-1}\left[\left\{A_l^N-\frac{N\zeta}{a_N}\left\{B_l^N-A_l^N\left[1-\frac{4a_N}{3N}\left(1-\frac{b_N}{a_N^2}\zeta\right)l|\xi|\right]\right\}\left(1-\frac{b_N}{a_N^2}\zeta\right)^{-1}\right.\right.\\
&+A_l^N\frac{N\zeta}{2a_N}\left[a_N^2\frac{4\eta^4(N^2-3N+3)}{9N(N-1)^3}\left(1-\frac{b_N}{a_N^2}\zeta\right)\right]+A_l^N\frac{N\zeta}{2a_N}\left[a_N\frac{\eta^2(N-2)}{N(N-1)^2}+\frac{3(N-2)^2}{4(N-1)}\left(1-\frac{b_N}{a_N^2}\zeta\right)^{-1}\right]\\
&+\frac{2N\zeta}{a_N}\left\{\frac{N}{2(N-1)}\left(1-\frac{b_N}{a_N^2}\zeta\right)-\frac{2a_N\eta^2}{3N}\left[\frac{N^2}{(N-1)^2}\right]\right\}\left\{C_l^N-\frac{1}{l}A_l^N\left[1-\frac{4a_N}{3N}\left(1-\frac{b_N}{a_N^2}\zeta\right)l|\xi|\right]\right\}\\
&\left.\left.-\frac{N\zeta}{a_N}\left\{D_l^N+K_l^N\left[1-\frac{4a_N}{3N}\left(1-\frac{b_N}{a_N^2}\zeta\right)l|\xi|\right]\right\}\left(1-\frac{b_N}{a_N^2}\zeta\right)^{-1}\right\}\exp\left[-\left(1-\frac{b_N}{a_N^2}\zeta\right)\left(\eta^2+\frac{4a_N}{3N}l|\xi|\right)\right]\right],
\end{aligned} \tag{106}$$

which is valid only to first order in ζ.

The canonical density (74) becomes

$$\begin{aligned}
\rho_c^*(\xi)&=L\rho_c(L\xi)=\frac{2\zeta c^2}{3\pi GM}\rho_c(L\xi)\\
&=\frac{4a_N}{3N}\exp\left[\frac{b_N}{a_N}\zeta\left(1-\frac{b_N}{a_N^2}\zeta\right)^{-1}\right]\sum_{l=1}^{N-1}\left\{A_l^N\left(1-\frac{b_N}{a_N^2}\zeta\right)\right.\\
&\left.+\frac{N\zeta}{a_N}\left(\frac{3}{8}\frac{(N-1)^2}{N}A_l^N-B_l^N-D_l^N\right)\right\}+\frac{N\zeta}{a_N}[A_l^N-K_l^N]\\
&\times\left[1-\frac{4a_N}{3N}\left(1-\frac{b_N}{a_N^2}\zeta\right)l|\xi|\right]\right\}\exp\left[-2(1-1/N)\right.\\
&\left.\times\left(1-\frac{b_N}{a_N^2}\zeta\right)l|\xi|\right].
\end{aligned} \tag{107}$$

We plot in Fig. 4 the nonrelativistic canonical single-particle

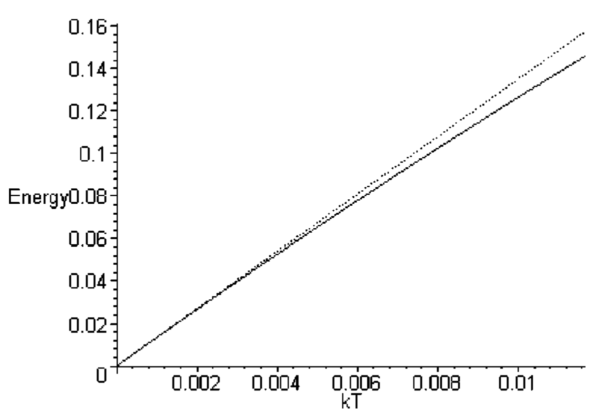

FIG. 3. Average energy $\langle\zeta\rangle$ for $N=10$ over the allowed range of kT.

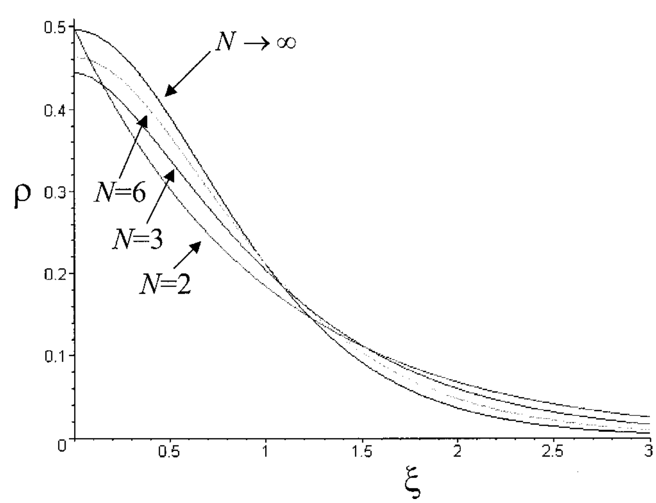

FIG. 4. The nonrelativistic canonical density function for various values of N.

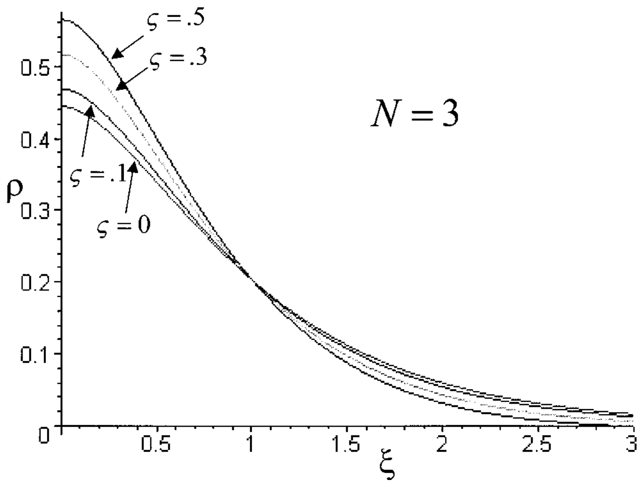

FIG. 5. The canonical density function for $N=3$ for various values of the relativistic parameter ζ. The nonrelativistic curve is labeled $\zeta=0$, and followed by curves for $\zeta=0.1$, $\zeta=0.3$, and $\zeta=0.5$, respectively. Here Eq. (105) yields $\zeta_{\max}\approx 0.43$ as the limit for which the relativistic expression is valid.

density function $\rho_c^*(\xi,\zeta=0)$ for various values of N, recovering the results of Rybicki [1]. With the exception of $N=2$, the central density grows with increasing N and the distribution becomes slightly more sharply peaked. It can be shown that as $N\to\infty$ the canonical density $\rho_c^*(\xi,\zeta=0)\to\frac{1}{2}\operatorname{sech}^2\xi$, and the single-particle distribution function approaches the isothermal solution of the Vlasov equation [1].

In Figs. 5–8 we plot the relativistic canonical function $\rho_c^*(\xi;\zeta)$ for differing values of N. As is clear from these figures, relativistic effects significantly enhance the central density by as much as 30% depending on the magnitude of ζ. Even for $\zeta=0.3$, the central density is larger than its value of 1/2 in the nonrelativistic large-N limit. While higher-order

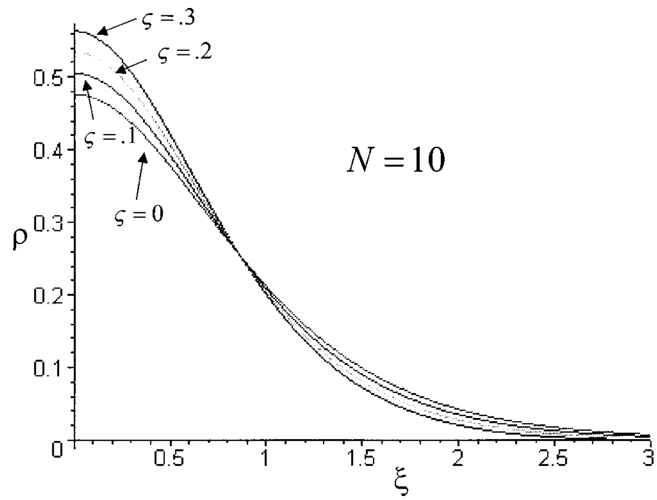

FIG. 7. The canonical density function for $N=10$ with curves that correspond to $\zeta=0$, 0.1, 0.2, 0.3, and $\zeta_{\max}\approx 0.15$.

corrections in ζ will modify this, the general trend is clear. The falloff of the relativistic density functions is also more rapid than in the nonrelativistic case.

Unfortunately there is no closed-form expression for the terms K_l^N and D_l^N and so it is not possible to evaluate an explicit expression for either $\rho_c^*(\xi)$ or $\rho_{mc}^*(\xi)$ in the large N limit. Instead these quantities must be computed using symbolic algebra; for $N>20$ this involves the factorization of thousands of terms, and computer memory limitations make this a prohibitive task. However, the large-N behavior should not be too different from the $N=20$ case, at least for small values of ζ. Figures 9 and 10 plot the density for different N at two different values of ζ.

The canonical momentum distribution function (75) is straightforwardly evaluated to be

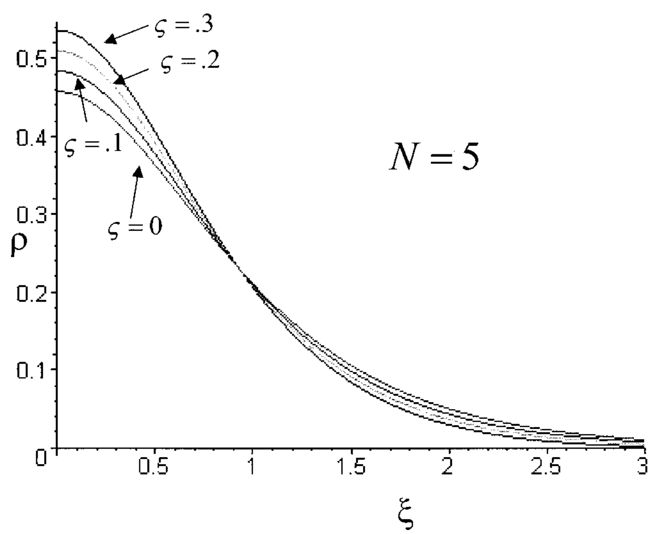

FIG. 6. The canonical density function for $N=5$ with curves that correspond to $\zeta=0$, 0.1, 0.2, 0.3, and $\zeta_{\max}\approx 0.29$.

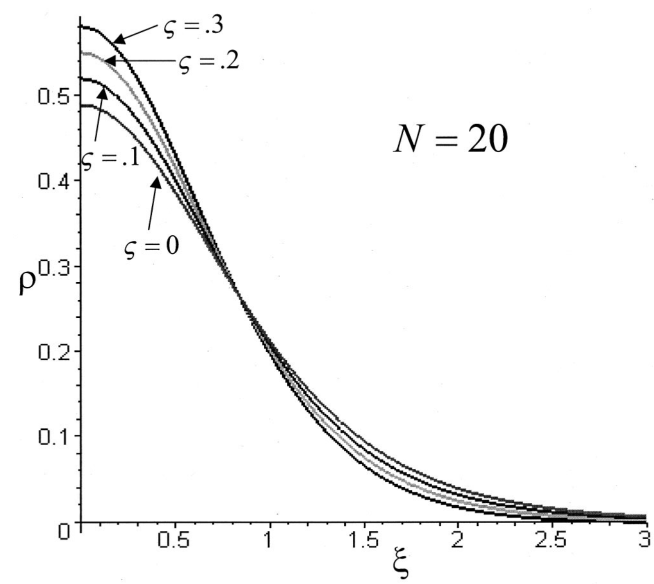

FIG. 8. The canonical density function for $N=20$ with curves that correspond to $\zeta=0$, 0.1, 0.2, 0.3, and $\zeta_{\max}\approx 0.15$.

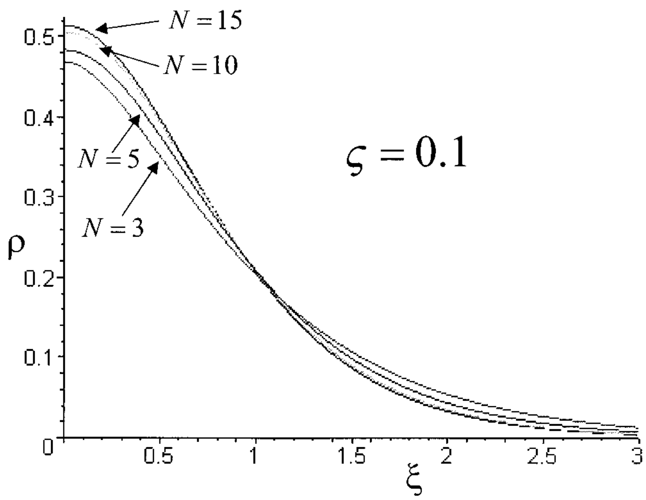

FIG. 9. The canonical density for different values of N at $\zeta=0.1$.

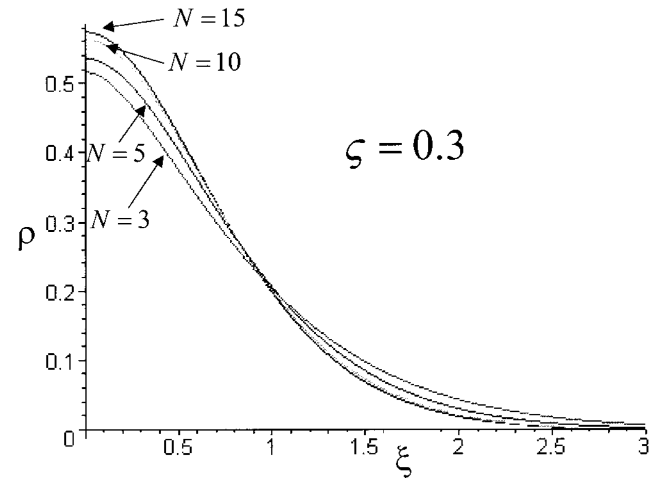

FIG. 10. The canonical density for different values of N at $\zeta=0.3$.

$$\begin{aligned}\vartheta^*_{cn}(\eta)&=mV\vartheta_{cn}(\sigma V\eta)\\&=m\sqrt{\frac{4\zeta c^2}{3}}\,\vartheta_{cn}(\sigma V\eta)\\&=\frac{1}{\sqrt{\pi}}\left(1-\frac{b_N}{a_N^2}\zeta\right)^{1/2}\exp\left[-\eta^2\left(1-\frac{b_N}{a_N^2}\zeta\right)\right]\\&\quad\times\left\{1+\frac{N\zeta}{a_N}\frac{(N^2-3N+3)}{2N(N-1)}\left(1-\frac{b_N}{a_N^2}\zeta\right)\eta^4\right.\\&\quad\times\frac{N\zeta}{a_N}\left[-\frac{\eta^2(4N^2-7N+6)}{2N(N-1)}+\frac{5N(N-1)+3}{8N(N-1)}\right.\\&\quad\left.\left.\times\left(1-\frac{b_N}{a_N^2}\zeta\right)^{-1}\right]\right\},\end{aligned}\tag{108}$$

where each of (107) and (108) are also valid to first order in ζ. Here we can easily take the large-N limit, which is

$$\vartheta^*_c(\eta)\to\frac{1}{\sqrt{\pi}}\left(1+\frac{\zeta}{54}[18\eta^4-9\eta^2-9-2\pi^2(2\eta^2-1)]\right)e^{-\eta^2},\tag{109}$$

to leading order in ζ.

We plot in Figs. 11–13 the behavior of the canonical momentum density as a function of the rescaled momentum η for differing values of N. The central momentum density increases with increasing ζ, and falls off more rapidly than in the nonrelativistic case.

However, for $\eta>2$, the momentum density grows relative to its nonrelativistic counterpart, overtaking this value for large enough η. The relative growth is exponential, although the overall momentum density is exponentially damped for any ζ. As N increases, the differences between the nonrelativistic and relativistic cases become less pronounced, although the basic features remain the same even in the limit that $N\to\infty$.

The microcanonical results are

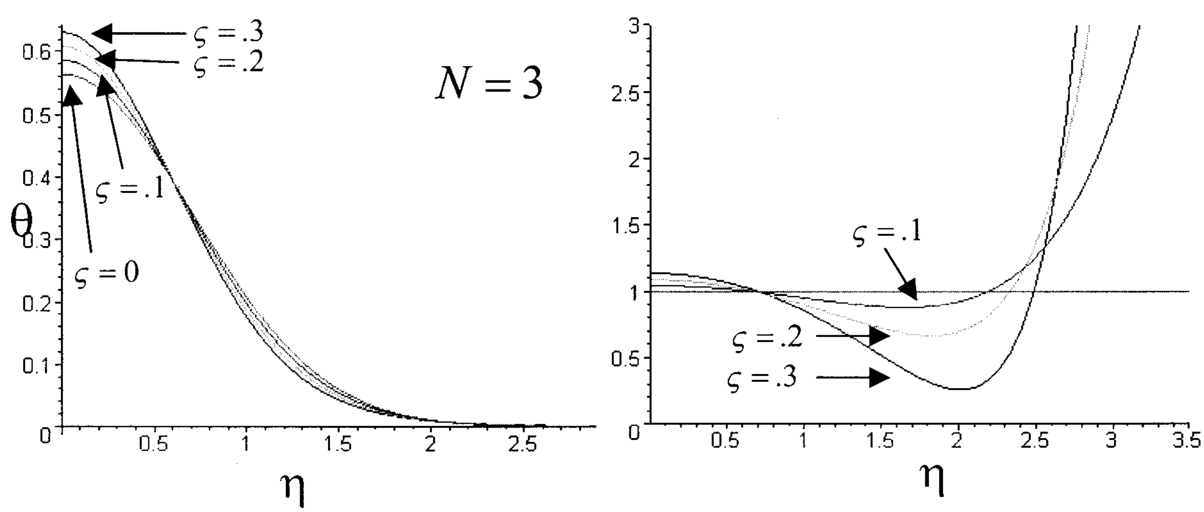

FIG. 11. The canonical momentum density as a function of η for $N=3$. On the left-hand side is the behavior of $\theta^*_c(\eta;\zeta)$ and on the right-hand side is its behavior relative to the nonrelativistic density $\theta^*_c(\eta;0)$. The curves correspond to $\zeta=0$, 0.1, 0.2, 0.3.

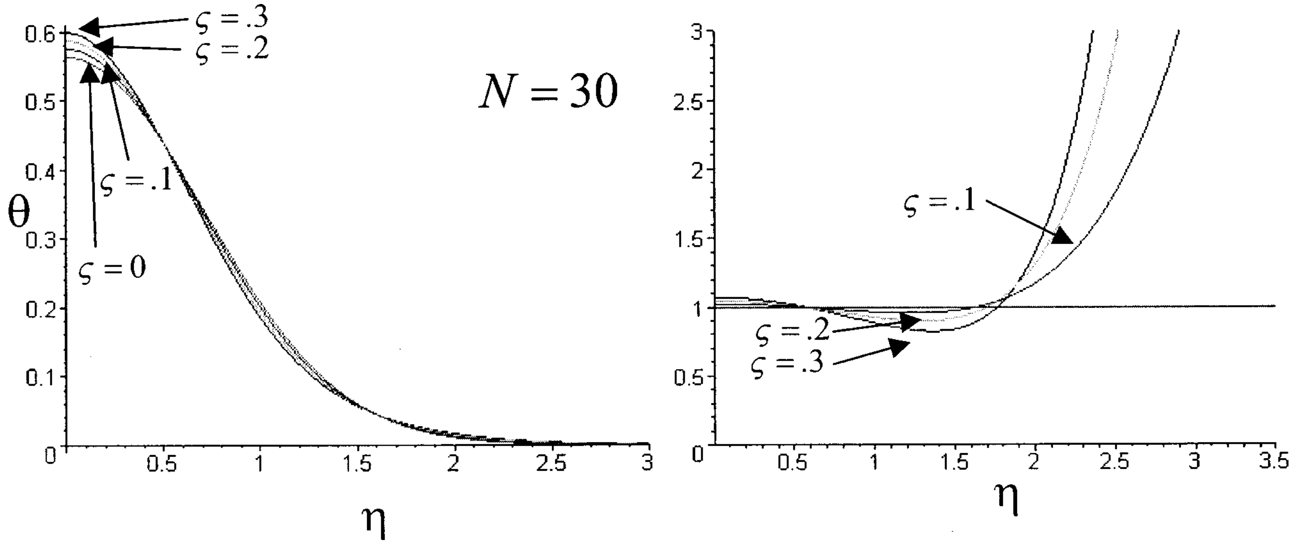

FIG. 12. The canonical momentum density as a function of η for $N=30$. Notation is as in Fig 11.

$$
\begin{aligned}
f'^{*R}_{mc}(\eta,\xi) &= \frac{2}{3N\pi G}\sqrt{\frac{4}{3}}(\zeta c^2)^{3/2} f'^{R}(mV\eta, L\xi) \\
&= \frac{2}{3N}\sqrt{\frac{2}{3\pi(N-1)}}\,\frac{\Gamma\left(\frac{3}{2}(N-1)\right)}{\Gamma\left(\frac{3}{2}(N-2)\right)}\exp\left[\frac{b_N}{a_N}\zeta\right]\sum_{l=1}^{N-1}\left\{A_l^N\left(1-\frac{2\eta^2}{3(N-1)}-\frac{4l|\xi|}{3N}\right)_+^{3N/2-4}\right. \\
&\quad +\zeta\eta^2\left\{A_l^N\left[\frac{(N-2)}{(N-1)^2}\right]-\left(C_l^N-\frac{1}{l}A_l^N\right)\left[\frac{4N^2}{3(N-1)^2}\right]\right\}\left(1-\frac{2\eta^2}{3(N-1)}-\frac{4l|\xi|}{3N}\right)_+^{3N/2-4} \\
&\quad +\left(\frac{3}{2}N-4\right)\zeta A_l^N\left[\frac{2\eta^4(N^2-3N+3)}{9(N-1)^3}-\frac{16N|\xi|\eta^2}{9(N-1)^2}\right]\left(1-\frac{2\eta^2}{3(N-1)}-\frac{4l|\xi|}{3N}\right)_+^{3N/2-5} \\
&\quad +\left\{\frac{4\zeta}{3}|\xi|\left[A_l^N\left(\frac{N}{(N-1)}-l\right)+lK_l^N\right]\right\}\left(1-\frac{2\eta^2}{3(N-1)}-\frac{4l|\xi|}{3N}\right)_+^{3N/2-4} \\
&\quad \left.+\frac{N\zeta}{\left(\frac{3}{2}N-3\right)}\left(A_l^N\left[\frac{3(N-2)^2}{8(N-1)}+1\right]-B_l^N+\frac{N\left(C_l^N-\frac{1}{l}A_l^N\right)}{(N-1)}-D_l^N-K_l^N\right)\left(1-\frac{2\eta^2}{3(N-1)}-\frac{4l|\xi|}{3N}\right)_+^{3N/2-3}\right\},
\end{aligned}
\tag{110}
$$

$$
\begin{aligned}
\rho^*_{mc}(\xi) &\equiv L\rho_{mc}(L\xi) \\
&= 2\exp\left[\frac{b_N}{a_N}\zeta\right]\sum_{l=1}^{N-1}\left\{\left(\frac{3N-5}{3N}\right)\left[A_l^N-\frac{4l|\xi|}{3}\zeta(A_l^N-K_l^N)\right]\left(1-\frac{4l|\xi|}{3N}\right)_+^{3N/2-7/2}\right. \\
&\quad \left.+\frac{2}{3}\zeta\left(1-\frac{4l|\xi|}{3N}\right)_+^{3N/2-5/2}\left[\left(\frac{3(N-1)^2}{8N}\right)A_l^N-(B_l^N-A_l^N)-(D_l^N+K_l^N)\right]\right\},
\end{aligned}
\tag{111}
$$

and

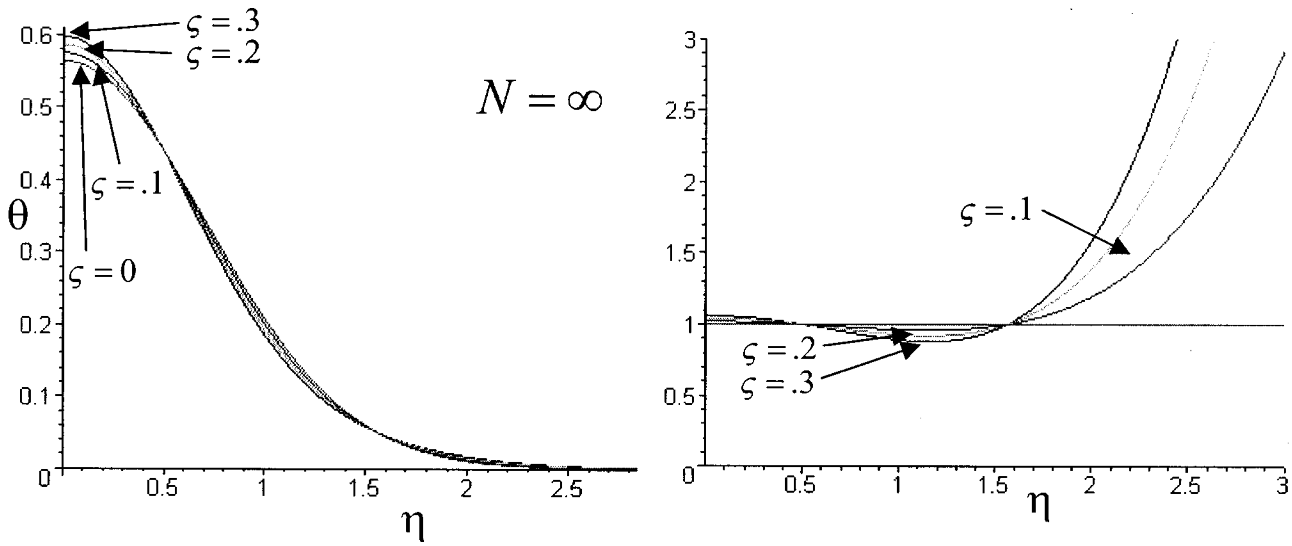

FIG. 13. The canonical momentum density as a function of η for $N=\infty$. Notation is as in Fig. 11.

$$\vartheta^*_{mc}(\eta)\equiv mV\vartheta_{mc}(mV\eta)$$

$$=\sqrt{\frac{2}{3\pi(N-1)}}\,\frac{\Gamma\left(\frac{3}{2}(N-1)\right)}{\Gamma\left(\frac{3}{2}N-2\right)}\exp\left[\frac{b_N}{a_N}\zeta\right]\Bigg\{\left[1-\left(\frac{\eta^2}{3}\left[\frac{8N^2-11N+6}{(N-1)^2}\right]\right)\zeta\right]\left(1-\frac{2\eta^2}{3(N-1)}\right)_+^{3N/2-3}$$
$$+\frac{\zeta(N-2)}{3}\left[\frac{\eta^4(N^2-3N+3)}{(N-1)^3}\right]\left(1-\frac{2\eta^2}{3(N-1)}\right)_+^{3N/2-4}$$
$$-\frac{2N\zeta}{(3N-4)}\left(\frac{(5N^2-20N+12)}{8(N-1)}+\sum_{s=1}^{N-1}\sum_{t=s+1}^{N-1}\frac{(t-s)}{t(N-s)}\right)\left(1-\frac{2\eta^2}{3(N-1)}\right)_+^{3N/2-2}\Bigg\}, \tag{112}$$

where each of Eqs. (110), (111), and (112) are valid to first order in ζ. As $N\to\infty$ we have

$$\vartheta^*_{mc}(\eta)\to\frac{1}{\sqrt{\pi}}\left(1+\frac{\zeta}{54}[18\eta^4-81\eta^2+27-2\pi^2(2\eta^2-1)]\right)e^{-\eta^2}. \tag{113}$$

Note that this differs from the canonical momentum density unless $\zeta=0$.

The microcanonical density function $\rho^*_{mc}(\xi;\zeta)$ for differing values of N is plotted in Figs. 14–17. For $N=3$, the microcanonical density is uniform until $\xi=9/8$, after which it falls linearly to zero. For $\xi<9/8$, at most two particles can contribute to the density; in this region relativistic effects enhance their contribution. However, for $\xi>9/8$, at most one particle can contribute, and relativistic effects suppress its contribution until $\xi=9/4$, after which the density vanishes. For larger N, as in the canonical case, relativistic effects significantly enhance the central density by as much as 30%, depending on the size of ζ. Their falloff is more rapid, and

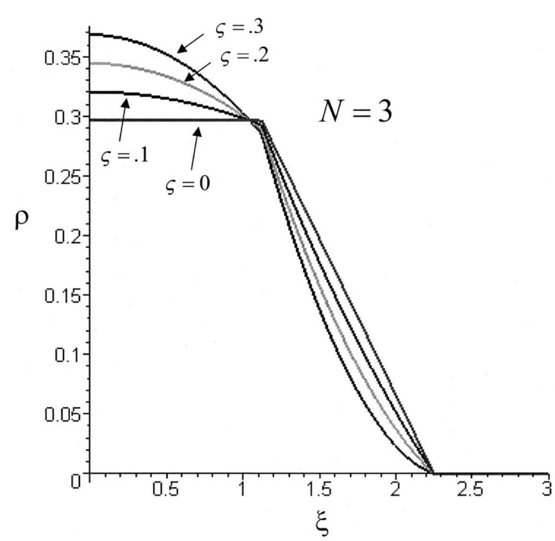

FIG. 14. The microcanonical density function for $N=3$ for various values of the relativistic parameter ζ. The nonrelativistic curve is labeled $\zeta=0$, followed by curves corresponding to $\zeta=0.1$, $\zeta=0.2$, and $\zeta=0.3$.

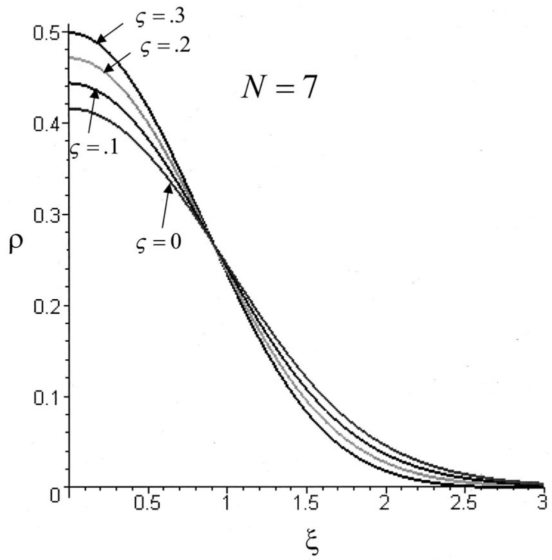

FIG. 15. The microcanonical density function for $N=7$ for various values of the relativistic paramter ζ. Notation is as in Fig. 14.

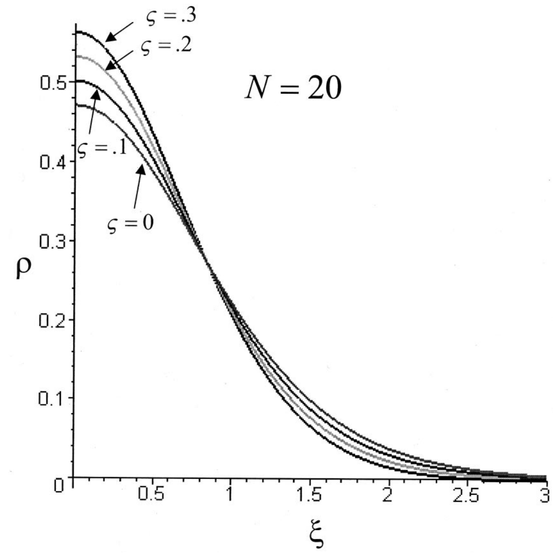

FIG. 17. The microcanonical density function for $N=20$ for various values of the relativistic parameter ζ. Notation is as in Fig. 14.

for sufficiently large ξ the nonrelativistic density distribution is larger. As N increases, the density becomes more sharply peaked and the contrasts between the nonrelativistic and relativistic cases become less pronounced.

In Figs. 18 and 19 we plot the microcanonical density distribution for increasing values of N and fixed ζ.

We plot the microcanonical momentum distributions in Figs. 20–22. The results are qualitatively similar to the canonical case, although the actual functional forms differ.

For small N, the relativistic approximation breaks down even for ζ as small as 0.3, and the momentum distribution function goes negative, as shown in Fig. 20. However, for larger N the momentum distribution is positive for all $\zeta \leq 0.3$, for example as in Fig. 21. The relativistic densities are more sharply peaked and for sufficiently large momentum parameter η are larger than their nonrelativistic counterparts.

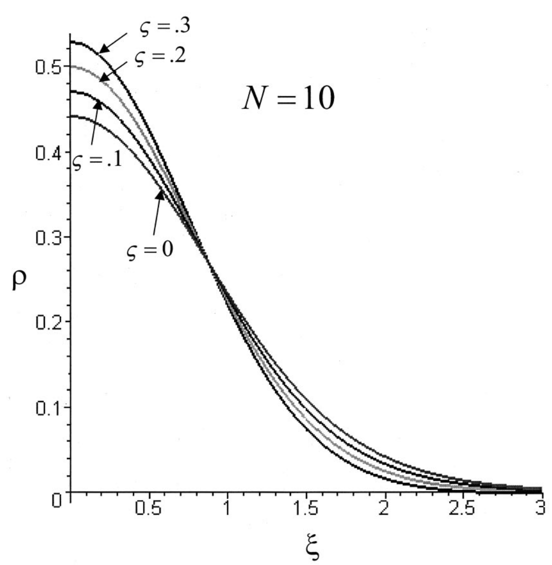

FIG. 16. The microcanonical density function for $N=10$ for various values of the relativistic parameter ζ. Notation is as in Fig. 14.

VII. CLOSING REMARKS

We have carried out an analysis of the statistical behavior of a ROGS to leading order in $1/c$. The qualitative behavior of the ROGS as compared to its nonrelativistic OGS counterpart [1] is clear. At a given energy, the ROGS temperature is smaller than the OGS temperature; relativistic effects cool the gas down. The one-particle distribution functions become more sharply peaked in each case with increasing N. For a given N, the ROGS density functions become more sharply peaked as the relativistic parameter ζ increases. This effect is commensurate with that observed in the exact two-body case, in which the maximal proper separation of a pair of particles is smaller in the relativistic system that in its non-

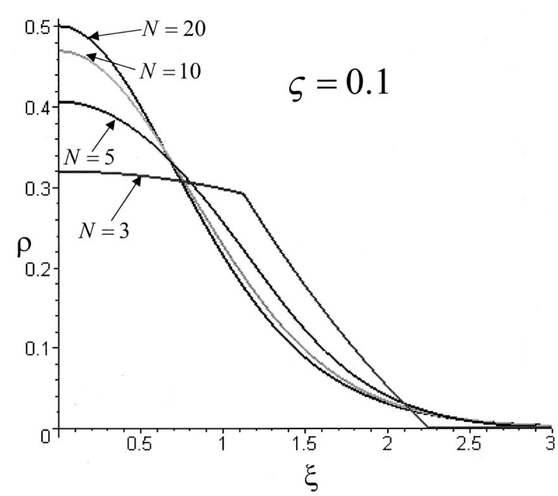

FIG. 18. The microcanonical density for different values of N at $\zeta=0.1$.

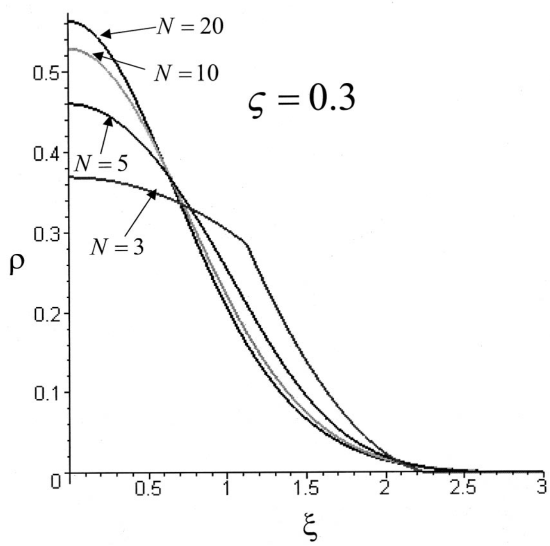

FIG. 19. The microcanonical density for different values of N at $\zeta=0.3$.

relativistic counterpart at the same energy [8,9]. Relativistic gravity introduces a quadratic spatial term in the potential, which makes the attractive force more powerful for larger distance at the same energy.

For both canonical and microcanonical distribution functions, $\rho_{\mathrm{OGS}}>\rho_{\mathrm{ROGS}}$ for sufficiently large position parameter ξ. However, for the momentum densities, this is not true. Although $\vartheta_{\mathrm{OGS}}>\vartheta_{\mathrm{ROGS}}$ for intermediate values of the momentum parameter η, once η becomes large enough this inequality is reversed. Again, this behavior is presumably due to quadratic character of the ROGS potential relative to its linear OGS counterpart in the former case, and from the p^4 corrections in the Hamiltonian (46) in the latter situation.

This work can be extended in several directions. It would be straightforward to extend these results to the charged and cosmological systems considered in Refs. [8,9] to see what effects these impose on the distribution functions. Extensions to unequal masses would also be interesting, although considerably more difficult. It would be very interesting to go beyond leading order in $1/c$ to investigate nonperturbative effects of the ROGS.

Further understanding the ROGS will undoubtedly require numerical experiments for various values of N. The equations of motion yield quartic (as opposed to quadratic) time dependence of the position variables (to leading order in $1/c$), and so can be straightforwardly (although somewhat tediously) integrated to investigate its equilibrium and equipartition properties. Work on this is in progress.

ACKNOWLEDGMENTS

This work was supported by the Natural Sciences and Engineering Research Council of Canada. We would like to thank T. Ohta for interesting discussions and correspondence.

APPENDIX

1. The Gaussian integrations

The following Gaussian integrations were used to obtain Eq. (51):

$$\begin{aligned}\int \frac{dk}{2\pi}\int d\mathbf{p}\exp\left(ik\sum_{a=1}^{N} p_a-\beta\sum_{a=1}^{N}\frac{p_a^2}{2m}\right) &= \int\frac{dk}{2\pi}\prod_{a=1}^{N}\int dp_a\, e^{-\beta(p_a^2/2m)+ikp_a}\\ &=\left(\frac{2m}{\beta}\right)^{N/2}\left(\frac{2m}{\beta}\right)^{-1/2}\int\frac{d\tilde{k}}{2\pi}\prod_{a=1}^{N}\int d\tilde{p}_a\, e^{-\tilde{p}_a^2+i\tilde{k}\tilde{p}_a}\\ &=\left(\frac{2m}{\beta}\right)^{(N-1)/2}\pi^{N/2}\int\frac{d\tilde{k}}{2\pi}\exp\left(-\frac{N\tilde{k}^2}{4}\right)\\ &=\frac{1}{\sqrt{N}}\left(\frac{2\pi m}{\beta}\right)^{(N-1)/2},\end{aligned}\tag{A1}$$

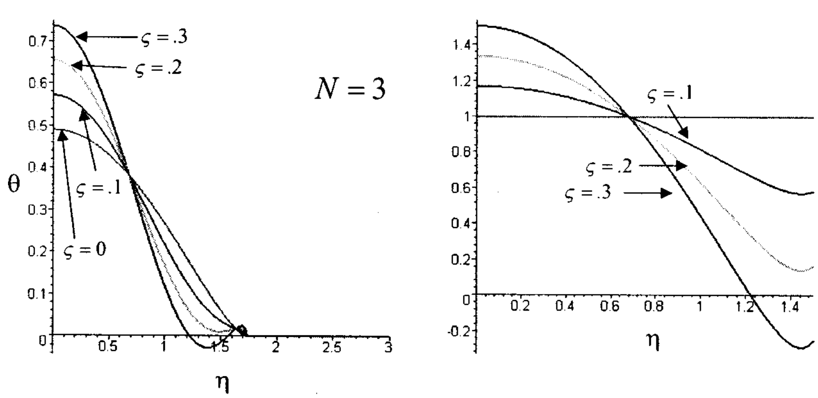

FIG. 20. The microcanonical momentum density as a function of η for $N=3$. On the left-hand side is the behavior of $\theta_{mc}^{*}(\eta;\zeta)$ and on the right-hand side is its behavior relative to the nonrelativistic density $\theta_c^{*}(\eta;0)$. The curves correspond to $\zeta=0$, 0.1, 0.2, and 0.3.

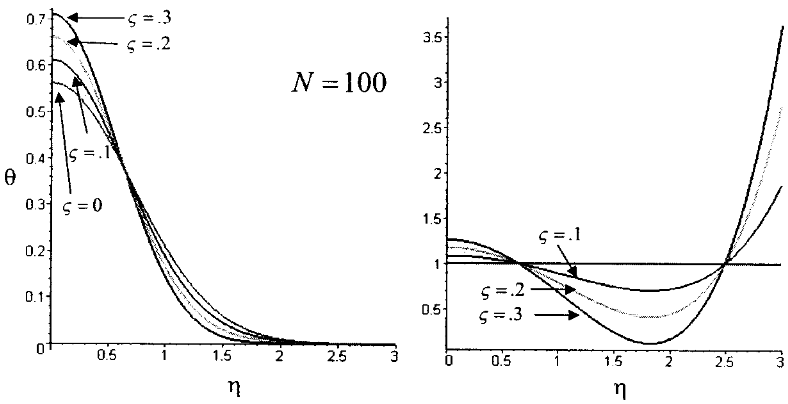

FIG. 21. The microcanonical momentum density as a function of η for $N=100$. Notation is as in Fig. 20.

$$
\begin{aligned}
&\int \frac{dk}{2\pi}\int d\mathbf{p}\, p_a^2 \exp\left(ik\sum_{b=1}^{N} p_b-\beta\sum_{b=1}^{N}\frac{p_b^2}{2m}\right)\\
&=\left(\frac{2m}{\beta}\right)^{(N+2)/2}\left(\frac{2m}{\beta}\right)^{-1/2}\int\frac{d\tilde{k}}{2\pi}\int d\tilde{p}_a\,\tilde{p}_a^2 e^{-\tilde{p}_a^2+i\tilde{k}\tilde{p}_a}\prod_{b\neq a}^{N}\int d\tilde{p}_b\, e^{-\tilde{p}_b^2+i\tilde{k}\tilde{p}_b}\\
&=\left(\frac{2m}{\beta}\right)^{(N+1)/2}\pi^{N/2}\int\frac{d\tilde{k}}{2\pi}\exp\left(-\frac{N\tilde{k}^2}{4}\right)\frac{1}{4}(2-\tilde{k}^2)\\
&=\pi^{N/2}\left(\frac{2m}{\beta}\right)^{(N+1)/2}\frac{N-1}{2\sqrt{\pi}N^{3/2}}=\frac{N-1}{2\pi N^{3/2}}\left(\frac{2\pi m}{\beta}\right)^{(N+1)/2},
\end{aligned}
\tag{A2}
$$

$$
\begin{aligned}
&\int \frac{dk}{2\pi}\int d\mathbf{p}\, p_b p_c \exp\left(ik\sum_{a=1}^{N} p_a-\beta\sum_{a=1}^{N}\frac{p_a^2}{2m}\right)\\
&=\left(\frac{2m}{\beta}\right)^{(N+2)/2}\left(\frac{2m}{\beta}\right)^{-1/2}\int\frac{d\tilde{k}}{2\pi}\int d\tilde{p}_b\,\tilde{p}_b e^{-p_b^2+i\tilde{k}p_b}\int d\tilde{p}_c\,\tilde{p}_c e^{-\tilde{p}_c^2+i\tilde{k}\tilde{p}_c}\prod_{a\neq b,c}^{N}\int d\tilde{p}_a\,\tilde{p}_a^2 e^{-\tilde{p}_a^2+i\tilde{k}\tilde{p}_a}\\
&=\left(\frac{2m}{\beta}\right)^{(N+1)/2}\pi^{N/2}\int\frac{d\tilde{k}}{2\pi}\exp\left(-\frac{N\tilde{k}^2}{4}\right)\left(\frac{-\tilde{k}^2}{4}\right)=-\frac{1}{2\pi N^{3/2}}\left(\frac{2\pi m}{\beta}\right)^{(N+1)/2},
\end{aligned}
\tag{A3}
$$

$$
\begin{aligned}
&\int \frac{dk}{2\pi}\int d\mathbf{p}\, p_b^4 \exp\left(ik\sum_{a=1}^{N} p_a-\beta\sum_{a=1}^{N}\frac{p_a^2}{2m}\right)=\left(\frac{2m}{\beta}\right)^{N+4)/2}\left(\frac{2m}{\beta}\right)^{-1/2}\int\frac{d\overline{k}}{2\pi}\int d\tilde{p}_a\,\tilde{p}_a^{-4}e^{-\tilde{p}_a^2+i\tilde{k}\overline{p}_a}\prod_{b\neq a}^{N}\int d\tilde{p}_b\, e^{-\tilde{p}_b^2+i\tilde{k}\tilde{p}_b}\\
&=\left(\frac{2m}{\beta}\right)^{(N+3)/2}\pi^{N/2}\int\frac{d\tilde{k}}{2\pi}\exp\left(-\frac{N\tilde{k}^2}{4}\right)\frac{1}{16}(\tilde{k}^4-12\tilde{k}^2+12)\\
&=\left(\frac{2m}{\beta}\right)^{(N+3)/2}\pi^{N/2}\frac{3(N-1)^2}{4\sqrt{\pi}N^{5/2}}=\frac{3(N-1)^2}{(2\pi)^2N^{5/2}}\left(\frac{2\pi m}{\beta}\right)^{(N+3)/2}.
\end{aligned}
\tag{A4}
$$

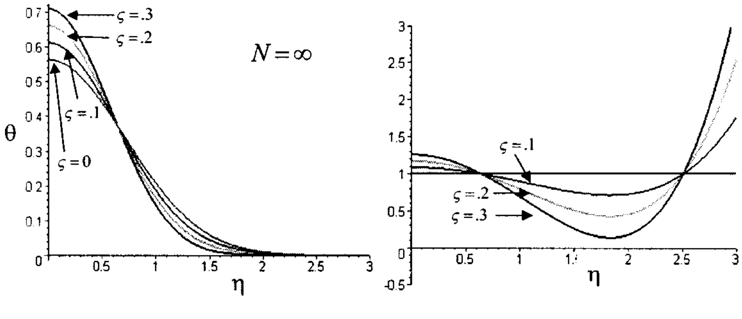

FIG. 22. The microcanonical momentum density as a function of η for $N=\infty$. Notation is as in Fig. 20.

2. The $\theta(p,z)$ term

This term is

$$\theta_{cn}(p,\mathbf{z})=\int \frac{dk}{2\pi}\int d\mathbf{p}\exp\left(ik\sum_{a=1}^{N} p_a-\beta\sum_{a=1}^{N}\frac{p_a^2}{2m}\right)\times\left(1-\frac{\beta}{c^2}H_R\right)\delta(p-p_n), \tag{A5}$$

where

$$H_R=-\lambda_1\sum_{a=1}^{N}\frac{p_a^4}{8m^3}+\pi G\lambda_2\sum_{a=1}^{N}\sum_{b=1}^{N}p_b^2|r_{ab}|-2\pi G\lambda_3\sum_{a>b}^{N}p_ap_b|r_{ab}|+\lambda_4(\pi G)^2\sum_{a=1}^{N}\sum_{b=1}^{N}\sum_{c=1}^{N}m^3[|r_{ab}||r_{ac}|-r_{ab}r_{ac}]. \tag{A6}$$

The classical and λ_4 part

Here we must compute

$$\int \frac{dk}{2\pi}\int d\mathbf{p}\exp\left(ik\sum_{a=1}^{N} p_a-\beta\sum_{a=1}^{N}\frac{p_a^2}{2m}\right)\times\delta(p-p_n)\left(1-\frac{\beta m^3}{c^2}\lambda_4(\pi G)^2\times\sum_{a=1}^{N}\sum_{b=1}^{N}\sum_{c=1}^{N}[|r_{ab}||r_{ac}|-r_{ab}r_{ac}]\right), \tag{A7}$$

of which the relevant part is

$$\int \frac{dk}{2\pi}\int d\mathbf{p}\exp\left(ik\sum_{a=1}^{N} p_\alpha-\beta\sum_{a=1}^{N}\frac{p_a^2}{2m}\right)\delta(p-p_n)$$
$$=\int\frac{dk}{2\pi}\exp\left(ikp-\frac{\beta p^2}{2m}\right)\int d\mathbf{p}\times\exp\left(ik\sum_{a\neq n}^{N} p_a-\beta\sum_{a\neq n}^{N}\frac{p_a^2}{2m}\right)$$
$$=\left(\frac{2m}{\beta}\right)^{(N-2)/2}\pi^{(N-1)/2}\int\frac{d\tilde{k}}{2\pi}\exp(i\tilde{k}\tilde{p}-\tilde{p}^2)\times\exp\left(-\frac{(N-1)\tilde{k}^2}{4}\right)$$
$$=\left(\frac{2\pi m}{\beta}\right)^{(N-2)/2}\frac{1}{\sqrt{N-1}}\exp\left(-\frac{\beta p^2N}{2m(N-1)}\right), \tag{A8}$$

and so

$$\int \frac{dk}{2\pi}\int d\mathbf{p}\exp\left(ik\sum_{a=1}^{N} p_a-\beta\sum_{a=1}^{N}\frac{p_a^2}{2m}\right)\times\delta(p-p_n)\left(1-\frac{\beta}{c^2}\lambda_4(\pi G)^2\times\sum_{1=1}^{N}\sum_{b=1}^{N}\sum_{c=1}^{N}m^3[|r_{ab}||r_{ac}|-r_{ab}r_{ac}]\right)$$
$$=\left(\frac{2\pi m}{\beta}\right)^{(N-2)/2}\frac{1}{\sqrt{N-1}}\exp\left(-\frac{\beta p^2N}{2m(N-1)}\right)\times\left(1-\frac{4\beta\lambda_4m^3(\pi G)^2}{c^2}\times\sum_{k=1}^{N-1}\sum_{l=k+1}^{N-1}(N-l)(l-k)ku_lu_k\right), \tag{A9}$$

where Eq. (57) was used.

3. The λ_1 part

Now we have

$$\int \frac{dk}{2\pi}\int d\mathbf{p}\exp\left(ik\sum_{a=1}^{N} p_a-\beta\sum_{a=1}^{N}\frac{p_a^2}{2m}\right)\delta(p-p_n)\left(-\frac{\beta}{c^2}\right)\times\left(-\lambda_1\sum_{c=1}^{N}\frac{p_c^4}{8m^3}\right)$$
$$=\frac{\lambda_1\beta}{8m^3c^2}\sum_{c=1}^{N}\int\frac{dk}{2\pi}\int d\mathbf{p}\,p_c^4\times\exp\left(ik\sum_{a=1}^{N} p_a-\beta\sum_{a=1}^{N}\frac{p_a^2}{2m}\right)\delta(p-p_n), \tag{A10}$$

and

$$\int \frac{dk}{2\pi}\int d\mathbf{p}\, p_c^4 \exp\left(ik\sum_{a=1}^{N} p_a-\beta\sum_{a=1}^{N}\frac{p_a^2}{2m}\right)\delta(p-p_n)$$

$$=\int \frac{dk}{2\pi}\exp\left(ikp-\frac{\beta p^2}{2m}\right)\int d\mathbf{p}\exp\left(ik\sum_{a\neq n}^{N} p_\alpha-\beta\sum_{a\neq n}^{N}\frac{p_a^2}{2m}\right)\begin{cases} p_c^4 & (c\neq n)\\ p^4 & (c=n)\end{cases}$$

$$=\left(\frac{2\pi m}{\beta}\right)^{(N+2)/2}\frac{1}{\pi^2}\exp\left(-\frac{\beta p^2 N}{2m(N-1)}\right)\begin{cases}\left(\frac{\beta^2p^4}{(2m)^2(N-1)^{9/2}}+\frac{3(N-2)\beta\rho^4}{2m(N-1)^{7/2}}+\frac{3(N-2)^2}{4(N-1)^{5/2}}\right) & (c\neq n)\\ \left(\frac{\beta^2p^4}{(2m)^2(N-1)^{1/2}}\right) & (c=n),\end{cases} \tag{A11}$$

which gives

$$\int \frac{dk}{2\pi}\int d\mathbf{p}\exp\left(ik\sum_{a=1}^{N} p_a-\beta\sum_{a=1}^{N}\frac{p_a^2}{2m}\right)\delta(p-p_n)\left(-\frac{\beta}{c^2}\right)\left(-\lambda_1\sum_{c=1}^{N}\frac{p_c^4}{8m^3}\right)$$

$$=\left(\frac{2\pi m}{\beta}\right)^{(N+2)/2}\frac{\lambda_1\beta}{8m^3\pi^2c^2}\exp\left(-\frac{\beta p^2N}{2m(N-1)}\right)$$

$$\times\left\{(N-1)\left(\frac{\beta^2p^4}{(2m)^2(N-1)^{9/2}}+\frac{3(N-2)\beta p^2}{2m(N-1)^{7/2}}+\frac{3(N-2)^2}{4(N-1)^{5/2}}\right)+\frac{\beta^2p^4}{(2m)^2(N-1)^{1/2}}\right\}$$

$$=\left(\frac{2\pi m}{\beta}\right)^{(N-2)/2}\frac{\exp\left(-\frac{\beta p^2N}{2m(N-1)}\right)}{\sqrt{N-1}}\frac{\lambda_1}{2\beta mc^2}\left(\frac{\beta^2p^4[1+(N-1)^3]}{(2m)^2(N-1)^3}+\frac{3(N-2)\beta p^2}{2m(N-1)^2}+\frac{3(N-2)^2}{4(N-1)}\right). \tag{A12}$$

a. The $\boldsymbol{\lambda_2}$ ***part***

Now we must do

$$\int \frac{dk}{2\pi}\int d\mathbf{p}\exp\left(ik\sum_{a=1}^{N} p_a-\beta\sum_{a=1}^{N}\frac{p_a^2}{2m}\right)\delta(p-p_n)\left(-\frac{\beta}{c^2}\right)$$

$$\times\left(+\pi G\lambda_2\sum_{a=1}^{N}\sum_{b=1}^{N}p_b^2|r_{ab}|\right)$$

$$=-\frac{\lambda_2\pi G\beta}{c^2}\sum_{c=1}^{N}\sum_{b=1}^{N}|r_{bc}|\int\frac{dk}{2\pi}\int d\mathbf{p}\,p_b^2$$

$$\times\exp\left(ik\sum_{a=1}^{N}p_a-\beta\sum_{a=1}^{N}\frac{p_a^2}{2m}\right)\delta(p-p_a), \tag{A13}$$

and the integrals are

$$\int \frac{dk}{2\pi}\int d\mathbf{p}\,p_c^2\exp\left(ik\sum_{a=1}^{N} p_a-\beta\sum_{a=1}^{N}\frac{p_a^2}{2m}\right)\delta(p-p_n)$$

$$=\int\frac{dk}{2\pi}\exp\left(ikp-\frac{\beta p^2}{2m}\right)$$

$$\times\int d\mathbf{p}\exp\left(ik\sum_{a\neq n}^{N}p_a-\beta\sum_{a\neq n}^{N}\frac{p_a^2}{2m}\right)\begin{cases}p_c^2 & (c\neq n)\\ p^2 & (c=n)\end{cases}$$

$$=\left(\frac{2\pi m}{\beta}\right)^{N/2}\frac{1}{\pi}\exp\left(-\frac{\beta p^2N}{2m(N-1)}\right)$$

$$\times\begin{cases}\left(\frac{\beta p^2}{2m(N-1)^{5/2}}+\frac{(N-2)}{2(N-1)^{3/2}}\right) & (c\neq n)\\ \left(\frac{\beta p^2}{2m(N-1)^{1/2}}\right) & (c=n),\end{cases} \tag{A14}$$

giving

$$\int \frac{dk}{2\pi}\int d\,\mathbf{p}\exp\left(ik\sum_{a=1}^{N} p_a-\beta\sum_{a=1}^{N}\frac{p_a^2}{2m}\right)\delta(p-p_n)\left(-\frac{\beta}{c^2}\right)\left(+\pi G\lambda_2\sum_{a=1}^{N}\sum_{b=1}^{N} p_b^2|r_{ab}|\right)$$
$$=-\frac{\lambda_2\pi G\beta}{c^2}\left(\frac{2\pi m}{\beta}\right)^{N/2}\frac{1}{\pi}\exp\left(-\frac{\beta p^2 N}{2m(N-1)}\right)\left[\sum_{c=1}^{N}\sum_{b\neq n}^{N}|r_{bc}|\left\{\frac{\beta p^2}{2m(N-1)^{5/2}}+\frac{(N-2)}{2(N-1)^{3/2}}\right\}+\sum_{c=1}^{N}|r_{cn}|\left(\frac{\beta p^2}{2m(N-1)^{1/2}}\right)\right]$$
$$=-\frac{2\lambda_2\pi G m}{c^2}\frac{1}{\sqrt{N-1}}\left(\frac{2\pi m}{\beta}\right)^{(N-2)/2}\exp\left(-\frac{\beta p^2 N}{2m(N-1)}\right)$$
$$\times\left[\sum_{c=1}^{N}\sum_{b\neq n}^{N}|r_{bc}|\left\{\frac{\beta p^2}{2m(N-1)^2}+\frac{(N-2)}{2(N-1)}\right\}+\sum_{c=1}^{N}|r_{cn}|\left(+\frac{\beta p^2}{2m}\right)\right]. \quad \text{(A15)}$$

Since $\Sigma_{b\neq n}^{N}|r_{cb}|=\Sigma_{b=1}^{N}|r_{cb}|-|r_{cn}|$, this becomes

$$\int \frac{dk}{2\pi}\int d\mathbf{p}\exp\left(ik\sum_{a=1}^{N} p_a-\beta\sum_{a=1}^{N}\frac{p_a^2}{2m}\right)\delta(p-p_n)\left(-\frac{\beta}{c^2}\right)\left(+\pi G\lambda_2\sum_{a=1}^{N}\sum_{b=1}^{N} p_b^2|r_{ab}|\right)$$
$$=-\frac{2\lambda_2\pi G m}{\sqrt{N-1}c^2}\left(\frac{2\pi m}{\beta}\right)^{(N-2)/2}\exp\left(-\frac{\beta p^2 N}{2m(N-1)}\right)\left\{\sum_{c=1}^{N}\sum_{b=1}^{N}|r_{bc}|\left\{\frac{\beta p^2}{2m(N-1)^2}+\frac{(N-2)}{2(N-1)}\right\}\right.$$
$$\left.+\sum_{c=1}^{N}|r_{cn}|\left(+\frac{\beta p^2}{2m}\left[\frac{N(N-2)}{(N-1)^2}\right]-\frac{(N-2)}{2(N-1)}\right)\right\}$$
$$=-\frac{2\lambda_2\pi G m}{\sqrt{N-1}c^2}\left(\frac{2\pi m}{\beta}\right)^{(N-2)/2}\exp\left(-\frac{\beta p^2 N}{2m(N-1)}\right)\left(2\sum_{l=1}^{N-1} l(N-l)u_l\left\{\frac{\beta p^2}{2m(N-1)^2}+\frac{(N-2)}{2(N-1)}\right\}\right.$$
$$\left.+\left(\sum_{s=1}^{n-1} s u_s+\sum_{s=n}^{N-1}(N-s)u_s\right)\left(+\frac{\beta p^2}{2m}\left[\frac{N(N-2)}{(N-1)^2}\right]-\frac{(N-2)}{2(N-1)}\right)\right). \quad \text{(A16)}$$

b. The λ_3 part

Now we must do

$$\int \frac{dk}{2\pi}\int d\mathbf{p}\exp\left(ik\sum_{a=1}^{N} p_a-\beta\sum_{a=1}^{N}\frac{p_a^2}{2m}\right)\delta(p-p_n)\left(-\frac{\beta}{c^2}\right)\left(-2\pi G\lambda_3\sum_{a>b}^{N} p_a p_b|r_{ab}|\right)$$
$$=\frac{\lambda_3\pi G\beta}{c^2}\int\frac{dk}{2\pi}\int d\mathbf{p}\exp\left(ik\sum_{a=1}^{N} p_a-\beta\sum_{a=1}^{N}\frac{p_a^2}{2m}\right)\delta(p-p_n)\left(\sum_{c\neq n}^{N}\sum_{b\neq n}^{N} p_c p_b|r_{cb}|+2p\sum_{c=1}^{N}|r_{cn}|p_c\right), \quad \text{(A17)}$$

where the integrations are

$$\int \frac{dk}{2\pi}\int d\mathbf{p}\, p_b p_c\exp\left(ik\sum_{a=1}^{N} p_a-\beta\sum_{a=1}^{N}\frac{p_a^2}{2m}\right)\delta(p-p_n)$$
$$=\left(\frac{2\pi m}{\beta}\right)^{N/2}\frac{1}{\pi}\exp\left(-\frac{\beta p^2 N}{2m(N-1)}\right)\begin{cases}\left(\dfrac{\beta p^2}{2m(N-1)^{5/2}}-\dfrac{1}{2(N-1)^{3/2}}\right) & (b\neq c\neq n)\\ \left(-\dfrac{\beta}{2m}\dfrac{p^2}{(N-1)^{3/2}}\right) & (c\neq n,\ \ b=n),\end{cases} \quad \text{(A18)}$$

giving

$$\int \frac{dk}{2\pi}\int d\mathbf{p}\exp\left(ik\sum_{a=1}^{N} p_a-\beta\sum_{a=1}^{N}\frac{p_a^2}{2m}\right)\delta(p-p_n)\left(-\frac{\beta}{c^2}\right)\left(-2\pi G\lambda_3\sum_{a>b}^{N} p_a p_b|r_{ab}|\right)$$
$$=\frac{\lambda_3\pi G\beta}{c^2}\left(\frac{2\pi m}{\beta}\right)^{N/2}\frac{1}{\pi}\exp\left(-\frac{\beta p^2 N}{2m(N-1)}\right)\left\{2\sum_{c=1}^{N}|r_{cn}|\left(-\frac{\beta}{2m}\frac{p^2}{(N-1)^{3/2}}\right)\right.$$
$$\left.+\sum_{c\neq n}^{N}\sum_{b\neq n}^{N}|r_{bc}|\left(\frac{\beta p^2}{2m(N-1)^{5/2}}-\frac{1}{2(N-1)^{3/2}}\right)\right\}. \tag{A19}$$

Using again $\Sigma_{b\neq n}^{N}|r_{cb}|=\Sigma_{b=1}^{N}|r_{cb}|-|r_{cn}|$, this becomes

$$\int \frac{dk}{2\pi}\int d\mathbf{p}\exp\left(ik\sum_{a=1}^{N} p_a-\beta\sum_{a=1}^{N}\frac{p_a^2}{2m}\right)\delta(p-p_n)\left(-\frac{\beta}{c^2}\right)\left(-2\pi G\lambda_3\sum_{a>b}^{N} p_a p_b|r_{ab}|\right)$$
$$=\frac{\lambda_3\pi G\beta}{c^2}\left(\frac{2\pi m}{\beta}\right)^{N/2}\frac{1}{\pi}\exp\left(-\frac{\beta p^2 N}{2m(N-1)}\right)\left\{2\sum_{c=1}^{N}|r_{cn}|\left(-\frac{\beta}{2m}\frac{p^2}{(N-1)^{3/2}}\right)+\left(\sum_{c=1}^{N}\sum_{b=1}^{N}|r_{bc}|-2\sum_{c=1}^{N}|r_{cn}|\right\}\right.$$
$$\left.\times\left(\frac{\beta p^2}{2m(N-1)^{5/2}}-\frac{1}{2(N-1)^{3/2}}\right)\right\}$$
$$=\frac{2\lambda_3\pi G m}{\sqrt{N-1}c^2}\left(\frac{2\pi m}{\beta}\right)^{(N-2)/2}\exp\left(-\frac{\beta p^2 N}{2m(N-1)}\right)\left\{\sum_{c=1}^{N}\sum_{b=1}^{N}|r_{bc}|\left(\frac{\beta p^2}{2m(N-1)^2}-\frac{1}{2(N-1)}\right)\right.$$
$$\left.+\sum_{c=1}^{N}|r_{cn}|\left(\frac{1}{(N-1)}-\frac{\beta p^2}{2m}\left[\frac{2N}{(N-1)^2}\right]\right)\right\}$$
$$=\frac{2\lambda_3 G m}{\sqrt{N-1}c^2}\left(\frac{2\pi m}{\beta}\right)^{(N-2)/2}\exp\left(-\frac{\beta p^2 N}{2m(N-1)}\right)\left\{2\sum_{l=1}^{N-1} l(N-1)u_l\left(\frac{\beta p^2}{2m(N-1)^2}-\frac{1}{2(N-1)}\right)\right.$$
$$\left.+\left(\sum_{s=1}^{n-1} s u_s+\sum_{s=n}^{N-1}(N-s)u_s\right)\left(\frac{1}{(N-1)}-\frac{\beta p^2}{2m}\left[\frac{2N}{(N-1)^2}\right]\right)\right\}. \tag{A20}$$

c. The full expression for $\theta_{cn}(p,z)$

From Eqs. (122), (125), (129), and (133) we have

$$\theta_{cn}(p,\mathbf{z})=\int\frac{dk}{2\pi}\int d\mathbf{p}\exp\left(ik\sum_{a=1}^{N} p_a-\beta\sum_{a=1}^{N}\frac{p_a^2}{2m}\right)\left(1-\frac{\beta}{c^2}H_R\right)\delta(p-p_n)$$
$$=\left(\frac{2\pi m}{\beta}\right)^{(N-2)/2}\frac{1}{\sqrt{N-1}}\exp\left(-\frac{\beta p^2 N}{2m(N-1)}\right)\left(1+\frac{\lambda_1}{2\beta m c^2}\left(\frac{\beta^2 p^4[1+(N-1)^3]}{(2m)^2(N-1)^3}+\frac{3(N-2)\beta p^2}{2m(N-1)^2}\right.\right.$$
$$\left.+\frac{3(N-2)^2}{4(N-1)}\right)+\frac{2\lambda_2\pi G m}{c^2}\left[-2\sum_{l=1}^{N-1} l(N-l)u_l\left\{\frac{\beta p^2}{2m(N-1)^2}+\frac{(N-2)}{2(N-1)}\right\}+\left(\sum_{s=1}^{n-1} s u_s+\sum_{s=n}^{N-1}(N-s)u_s\right)\right.$$
$$\left.\times\left(\frac{(N-2)}{2(N-1)}-\frac{\beta p^2}{2m}\left[\frac{N(N-2)}{(N-1)^2}\right]\right)\right]+\frac{2\lambda_3\pi G m}{c^2}\left\{2\sum_{l=1}^{N-1} l(N-l)u_l\left(\frac{\beta p^2}{2m(N-1)^2}-\frac{1}{2(N-1)}\right)\right.$$
$$\left.+\left(\sum_{s=1}^{n-1} s u_s+\sum_{s=n}^{N-1}(N-s)u_s\right)\left(\frac{1}{(N-1)}-\frac{\beta p^2}{2m}\left[\frac{2N}{(N-1)^2}\right]\right)\right\}$$
$$\left.-\frac{4\beta m^3}{c^2}\lambda_4(\pi G)^2\sum_{k=1}^{N-1}\sum_{l=k+1}^{N-1}(N-l)(l-k)k u_l u_k\right), \tag{A21}$$

or alternatively

$$\theta_{cn}(p,\mathbf{z})=\left(\frac{2\pi m}{\beta}\right)^{(N-2)/2}\frac{1}{\sqrt{N-1}}\exp\left(-\frac{\beta p^2 N}{2m(N-1)}\right)\Bigg[1+\frac{\lambda_1}{2\beta mc^2}\left(\frac{\beta^2p^4[1+(N-1)^3]}{(2m)^2(N-1)^3}+\frac{3(N-2)\beta p^2}{2m(N-1)^2}+\frac{3(N-2)^2}{4(N-1)}\right)$$
$$-\frac{4\beta m}{c^2}\lambda_4(\pi Gm)^2\sum_{k=1}^{N-1}\sum_{l=k+1}^{N-1}(N-l)(l-k)ku_lu_k+\frac{2\pi Gm}{c^2}\sum_{l=1}^{N-1}l(N-l)u_l$$
$$\times\left(\frac{(2\lambda_3-2\lambda_2)\beta p^2}{2m(N-1)^2}-\frac{[2\lambda_3+2(N-2)\lambda_2]}{2(N-1)}\right)+\frac{2[2\lambda_3+(N-2)\lambda_2]\pi Gm}{c^2}\left(\sum_{s=1}^{n-1}su_s+\sum_{s=n}^{N-1}(N-s)u_s\right)$$
$$\times\left(\frac{N-1-\frac{N\beta p^2}{m}}{2(N-1)^2}\right)\Bigg]. \tag{A22}$$

When the λ's are all equal to unity, this is

$$\theta_{cn}(p,\mathbf{u})=\frac{1}{\sqrt{N-1}}\exp\left(-\frac{N\beta p^2}{2m(N-1)}\right)\left(\frac{2\pi m}{\beta}\right)^{(N-2)/2}\Bigg\{\left(1-\frac{4\beta m(\pi Gm)^2}{c^2}\sum_{k=1}^{N-1}\sum_{l=k+1}^{N-1}(N-l)(l-k)ku_lu_k\right)$$
$$+\frac{1}{2\beta mc^2}\left[\frac{\beta^2p^4N(N^2-3N+3)}{(2m)^2(N-1)^3}+\frac{3\beta p^2(N-2)}{2m(N-1)^2}+\frac{3(N-2)^2}{4(N-1)}\right]+\left(\frac{2\pi mG}{c^2}\right)$$
$$\times\left[-\sum_{l=1}^{N-1}l(N-l)u_l-\left(\frac{N^2\beta p^2}{2m(N-1)^2}-\frac{N}{2(N-1)}\right)\left(\sum_{s=1}^{n-1}su_s+\sum_{s=n}^{N-1}(N-s)u_s\right)\right]\Bigg\}. \tag{A23}$$

4. An evaluation of $f_{cn}(p,z)$

Now consider Eq. (65), which can be rewritten as

$$f^R_{cn}(p,z)=\frac{e^{-\beta Mc^2}}{\mathcal{Z}}\int\frac{dk}{2\pi}\int d\mathbf{u}$$
$$\times\exp\left(-ikz-\lambda\beta\sum_{l=1}^{N-1}(C_l+i\alpha D_{nl})u_l\right)$$
$$\times\theta_{cn}(p,\mathbf{u}), \tag{A24}$$

where

$$\lambda=2\pi Gm^2,\quad \alpha=\frac{k}{N\beta\lambda}. \tag{A25}$$

The integrals are now

$$I_n^0=\int d\mathbf{n}\exp\left(-\lambda\beta\sum_{l=1}^{N-1}(C_l+i\alpha D_{nl})u_l\right)$$
$$=\frac{1}{(2\pi G\beta m^2)^{N-1}}\prod_{l=1}^{N-1}\frac{1}{C_l+i\alpha D_{nl}}, \tag{A26}$$

$$I^1_{n,s}=\int d\mathbf{u}\exp\left(-\lambda\beta\sum_{l=1}^{N-1}(C_l+i\alpha D_{nl})u_l\right)u_s$$
$$=\frac{1}{(2\pi G\beta m^2)^N}\frac{1}{C_s+i\alpha D_{ns}}\prod_{l=1}^{N-1}\frac{1}{C_l+i\alpha D_{nl}}, \tag{A27}$$

$$I^2_{n,s,t}=\int d\mathbf{u}\exp\left(-\lambda\beta\sum_{l=1}^{N-1}(C_l+i\alpha D_{nl})u_l\right)u_su_t$$
$$=\frac{1}{(2\pi G\beta m^2)^{N+1}}\frac{1}{C_s+i\alpha D_{ns}}\frac{1}{C_i+i\alpha D_{nt}}$$
$$\times\prod_{l=1}^{N-1}\frac{1}{C_l+i\alpha D_{nl}}, \tag{A28}$$

and we must divide by N and sum over all values of n to obtain the correct result.

For example,

$$\begin{aligned}\frac{1}{N}\sum_{n=1}^{N} I_n^0 &= \frac{1}{N(2\pi G\beta m^2)^{N-1}}\sum_{n=1}^{N}\left[\prod_{l=1}^{n-1}\frac{1}{l(N-l-i\alpha)}\prod_{l=n}^{N-1}\frac{1}{(N-l)(l+i\alpha)}\right]\\ &= \frac{1}{N(2\pi G\beta m^2)^{N-1}}\sum_{n=1}^{N}\frac{1}{(n-1)!(N-n)!}\frac{\Gamma(N-n+1-i\alpha)}{\Gamma(N-i\alpha)}\frac{\Gamma(n+i\alpha)}{\Gamma(N+i\alpha)}\\ &= \frac{1}{\Gamma(N-i\alpha)\Gamma(N+i\alpha)(2\pi G\beta m^2)^{N-1}}\sum_{n=1}^{N}\frac{(N-1)!}{(n-1)!(N-n)!}\int_0^1 dw\, w^{N-n-i\alpha}(1-w)^{n+i\alpha-1}\\ &= \frac{1}{\Gamma(N-i\alpha)\Gamma(N+i\alpha)(2\pi G\beta m^2)^{N-1}}\int_0^1 dw\, w^{-i\alpha}(1-w)^{i\alpha}(w+1-w)^{N-1}\\ &= \frac{1}{(2\pi G\beta m^2)^{N-1}}\frac{\Gamma(1-i\alpha)}{\Gamma(N-i\alpha)}\frac{\Gamma(1+i\alpha)}{\Gamma(N+i\alpha)}. \end{aligned} \tag{A29}$$

This function has single poles at $\alpha=in$, where n is an integer taking its values between $-N$ and N, except for $N=0$. Its residues are

$$\mathrm{Res}\left[\frac{\Gamma(1-i\alpha)}{\Gamma(N-i\alpha)}\frac{\Gamma(1+i\alpha)}{\Gamma(N+i\alpha)}\right]_{\alpha=in} = \frac{in(-1)^n}{\Gamma(N-n)\Gamma(N+n)}, \tag{A30}$$

yielding

$$\begin{aligned}\frac{1}{N}\sum_{n=1}^{N}\int\frac{dk}{2\pi}\int d\mathbf{u}\exp\left(-ikz-\lambda\beta\sum_{l=1}^{N-1}(C_l+i\alpha D_{nl})u_l\right)\\ = \frac{2N\beta\pi G m^2}{(2\pi G\beta m^2)^{N-1}[\Gamma(N)]^2}\sum_{l=1}^{N-1}A_l^N\exp[-2N(\beta\pi G m^2)l|z|], \end{aligned}\tag{A31}$$

for the leading nonrelativistic term, where

$$A_n^N = \frac{n(-1)^{n+1}[\Gamma(N)]^2}{\Gamma(N-n)\Gamma(N+n)}. \tag{A32}$$

The remaining integrals are somewhat more difficult. We obtain

$$\begin{aligned}\frac{1}{N}\sum_{n=1}^{N}\int d\mathbf{u}\exp\left(-ikz-\lambda\beta\sum_{l=1}^{N-1}(C_l+i\alpha D_{nl})u_l\right)\sum_{s=1}^{N-1}s(N-s)u_s\\ = -\frac{1}{\lambda\beta N}\sum_{n=1}^{N}\frac{d}{d\sigma}\left[\int d\mathbf{u}\exp\left(-\lambda\beta\sum_{l=1}^{N-1}(\sigma C_l+i\alpha D_{nl})u_l\right)\right]_{\sigma=1}\\ = -\frac{1}{\lambda\beta N}\frac{d}{d\sigma}\left[\sum_{n=1}^{N}\frac{1}{(\lambda\beta\sigma)^{N-1}}\prod_{l=1}^{N-1}\frac{1}{C_l+i(\alpha/\sigma)D_{nl}}\right]_{\sigma=1}\\ = -\frac{d}{d\sigma}\left[\frac{\sigma^{1-N}}{(\lambda\beta)^N}\frac{\Gamma(1-i\alpha/\sigma)}{\Gamma(N-i\alpha/\sigma)}\frac{\Gamma(1+i\alpha/\sigma)}{\Gamma(N+i\alpha/\sigma)}\right]_{\sigma=1}\\ = \frac{1}{(2\pi G\beta m^2)^N}\frac{\Gamma(1-i\alpha)}{(N-i\alpha)}\frac{\Gamma(1+i\alpha)}{\Gamma(N+i\alpha)}\{N-1+i\alpha[\Psi(N-i\alpha)-\Psi(1-i\alpha)\quad-\Psi(N+i\alpha)+\Psi(1+i\alpha)]\},\end{aligned}\tag{A33}$$

where $\Psi(x)=(d/dx)\ln\Gamma(x)$ is the digamma function. This function on the right-hand side of Eq. (A33) has a combination of single and double poles, each located at $\alpha=in$, where n is an integer taking its values between $-N$ and N, except for $N=0$. Writing

$$\Psi(N-i\alpha)-\Psi(1-i\alpha)=\sum_{s=1}^{N-1}\frac{1}{s-i\alpha},$$

we have the last line of Eq. (A33) proportional to

$$\frac{\Gamma(1-i\alpha)}{\Gamma(N-i\alpha)}\frac{\Gamma(1+i\alpha)}{\Gamma(N+i\alpha)}\left[N-1+i\alpha\sum_{s=1}^{N-1}\left(\frac{1}{s-i\alpha}-\frac{1}{s+i\alpha}\right)\right]$$
$$=\frac{\Gamma(1-i\alpha)}{\Gamma(N-i\alpha)}\frac{\Gamma(1+i\alpha)}{\Gamma(N+i\alpha)}\times\left[N-3-2\alpha^2\sum_{s\neq n}^{N-1}\left(\frac{1}{s^2+\alpha^2}\right)+\left\{\frac{2n^2}{\alpha^2+n^2}\right\}\right],$$

where the term in curly brackets contains the double pole at $\alpha=in$. Hence

$$\frac{1}{N}\sum_{n=1}^{N}\int\frac{dk}{2\pi}\int d\mathbf{u}\times\exp\left(-ikz-\lambda\beta\sum_{l=1}^{N-1}(C_l+i\alpha D_{nl})u_l\right)\times\sum_{s=1}^{N-1}s(N-s)u_s$$
$$=\frac{N\beta\lambda}{(\lambda\beta)^N}\int\frac{d\alpha}{2\pi}e^{-i\alpha N\beta\lambda_z}\frac{\Gamma(1-i\alpha)}{\Gamma(N-i\alpha)}\frac{\Gamma(1+i\alpha)}{\Gamma(N+i\alpha)}\times\left[N-3-2\alpha^2\sum_{s\neq n}^{N-1}\left(\frac{1}{s^2+\alpha^2}\right)+\left\{\frac{2n^2}{\alpha^2+n^2}\right\}\right].\tag{A34}$$

The residue from the double pole at $\alpha=in$ is

$$-i\frac{d}{dx}\left[\frac{2n^2x(-1)^n\exp(N\beta\lambda xz)}{\Gamma(N-x)\Gamma(N+x)(x+n)}\right]_{x=n}$$
$$=\frac{-in(-1)^n\exp(N\beta\lambda nz)}{2\Gamma(N-n)\Gamma(N+n)}\times\{1+2N\beta\lambda nz+2n[\Psi(N-n)-\Psi(N+n)]\},\tag{A35}$$

provided $z<0$, and so the total residue at $\alpha=in$ from Eq. (A34) is

$$\frac{in(-1)^n\exp(N\beta\lambda nz)}{\Gamma(N-n)\Gamma(N+n)}\left[N-\frac{5}{2}+2n^2\sum_{s\neq n}^{N-1}\left(\frac{l}{s^2-n^2}\right)+n[\Psi(N+n)-\Psi(N-n)]-(1+N\beta\lambda nz)\right]$$
$$\equiv\frac{-i\exp(N\beta\lambda nz)}{[\Gamma(N)]^2}[B_n^N-A_n^N(1+N\beta\lambda nz)],\tag{A36}$$

giving finally

$$\frac{1}{N}\sum_{n=1}^{N}\int\frac{dk}{2\pi}\int d\mathbf{u}\exp\left(-ikz-\lambda\beta\sum_{l=1}^{N-1}(C_l+i\alpha D_{nl})u_l\right)\sum_{s=1}^{N-1}s(N-s)u_s$$
$$=\frac{2N\beta\pi Gm^2}{(2\pi G\beta m^2)^N[\Gamma(N)]^2}\sum_{l=1}^{N-1}\{B_l^N-A_l^N[1-2N(\beta\pi Gm^2)l|z|]\}\exp[-2N(\beta\pi Gm^2)l|z|],\tag{A37}$$

where

$$B_n^N=\frac{n(-1)^{n+1}[\Gamma(N)]|^2}{\Gamma(N-n)\Gamma(N+n)}\left[N-\frac{5}{2}+2n^2\sum_{s\neq n}^{N-1}\left(\frac{1}{s^2-n^2}\right)+n[\Psi(N+n)-\Psi(N-n)]\right].\tag{A38}$$

Using the relation

$$\sum_{c=1}^{N}|r_{cn}|=\sum_{l=1}^{n-1}lu_l+\sum_{l=n}^{N-1}(N-l)u_l,\tag{A39}$$

we find

$$\frac{1}{N}\sum_{n=1}^{N}\int d\mathbf{u}\exp\left(-\lambda\beta\sum_{l=1}^{N-1}(C_l+i\alpha D_{nl})u_l\right)\sum_{s=1}^{N}|r_{sn}|$$
$$=\frac{1}{N}\sum_{n=1}^{N}\left(\sum_{s=1}^{n-1}sI_{n,s}^1+\sum_{s=n}^{N-1}(N-s)I_{n,s}^1\right)$$
$$=\frac{1}{N(2\pi G\beta m^2)^N}\sum_{n=1}^{N}\left[\sum_{s=1}^{n-1}\frac{1}{(N-s-i\alpha)}+\sum_{s=n}^{N-1}\frac{1}{(s+i\alpha)}\right]\left(\prod_{l=1}^{N-1}\frac{1}{C_l+i\alpha D_{nl}}\right)$$

$$= \frac{1}{N(2\pi G\beta m^2)^N}\sum_{n=1}^{N}\left\{[\Psi(1-N+i\alpha)-\Psi(n-N+i\alpha)+\Psi(N+i\alpha)-\Psi(n+i\alpha)]\right.$$
$$\left.\times\left(\frac{1}{(n-1)!(N-n)!}\frac{\Gamma(N-n+1-i\alpha)}{\Gamma(N-i\alpha)}\frac{\Gamma(n+i\alpha)}{\Gamma(N+i\alpha)}\right)\right\}$$
$$= \frac{1}{N(2\pi G\beta m^2)^N}\sum_{n=1}^{N}\left\{[\Psi(N-i\alpha)+\Psi(N+i\alpha)-\Psi(N-n+1-i\alpha)-\Psi(n+i\alpha)]\right.$$
$$\left.\times\left(\frac{1}{(n-1)!(N-n)!}\frac{\Gamma(N-n+1-i\alpha)}{\Gamma(N-i\alpha)}\frac{\Gamma(n+i\alpha)}{\Gamma(N+i\alpha)}\right)\right\}$$
$$= \frac{1}{N(2\pi G\beta m^2)^N}\sum_{n=1}^{N}\left[[\Psi(N-i\alpha)+\Psi(N+i\alpha)]\left(\frac{1}{(n-1)!(N-n)!}\frac{\Gamma(N-n+1-i\alpha)}{\Gamma(N-i\alpha)}\frac{\Gamma(n+i\alpha)}{\Gamma(N+i\alpha)}\right)\right.$$
$$\left.-\frac{d}{d\sigma}\left(\frac{1}{(n-1)!(N-n)!}\frac{\Gamma(N-n+1-i\alpha+\sigma)}{\Gamma(N-i\alpha)}\frac{\Gamma(n+i\alpha+\sigma)}{\Gamma(N+i\alpha)}\right)\right]_{\sigma=0}$$
$$=\frac{1}{(2\pi G\beta m^2)^N}[\Psi(N-i\alpha)+\Psi(N+i\alpha)]\frac{\Gamma(1-i\alpha)}{\Gamma(N-i\alpha)}\frac{\Gamma(1+i\alpha)}{\Gamma(N+i\alpha)}$$
$$-\left[\frac{1}{\Gamma(N+1)}\frac{d}{d\sigma}\left(\frac{\Gamma(N+1+2\sigma)\Gamma(1-i\alpha+\sigma+\sigma)\Gamma(1+i\alpha+\sigma)}{\Gamma(2+2\sigma)\Gamma(N-i\alpha)\Gamma(N+i\alpha)}\right)\right]_{\sigma=0}$$
$$=\frac{1}{(2\pi G\beta m^2)^N}\frac{\Gamma(1-i\alpha)}{\Gamma(N-i\alpha)}\frac{\Gamma(1+i\alpha)}{\Gamma(N+i\alpha)}[\Psi(N-i\alpha)+\Psi(N+i\alpha)-\Psi(1-i\alpha)-\Psi(1+i\alpha)-2\Psi(N+1)+2\Psi(2)]$$
$$=\frac{1}{(2\pi G\beta m^2)^N}\frac{\Gamma(1-i\alpha)}{\Gamma(N-i\alpha)}\frac{\Gamma(1+i\alpha)}{\Gamma(N+i\alpha)}\left[\sum_{s=1}^{N-1}\frac{1}{s-i\alpha}+\sum_{s=1}^{N-1}\frac{1}{s+i\alpha}-2\sum_{s=1}^{N-1}\frac{1}{s+1}\right]$$
$$=\frac{1}{(2\pi G\beta m^2)^N}\frac{\Gamma(1-i\alpha)}{\Gamma(N-i\alpha)}\frac{\Gamma(1+i\alpha)}{\Gamma(N+i\alpha)}\left[\sum_{s\neq n}^{N-1}\frac{1}{s-i\alpha}+\sum_{s\neq n}^{N-1}\frac{1}{s+i\alpha}-2\sum_{s=1}^{N-1}\frac{1}{s+1}+\left\{\frac{2n}{\alpha^2+n^2}\right\}\right], \tag{A40}$$

where the curly bracket contains the double pole at $\alpha=in$, and all other poles are in the same locations as before.

So we obtain

$$\frac{1}{N}\sum_{n=1}^{N}\int\frac{dk}{2\pi}\int d\mathbf{u}\exp\left(-ikz-\lambda\beta\sum_{l=1}^{N-1}(C_l+i\alpha D_{nl})u_l\right)\left(\sum_{l=1}^{n-1}lu_l+\sum_{l=n}^{N-1}(N-l)u_l\right)$$
$$=\frac{N\beta\lambda}{(2\pi G\beta m^2)^N}\int\frac{d\alpha}{2\pi}e^{-i\alpha N\beta\lambda z}\frac{\Gamma(1-i\alpha)}{\Gamma(N-i\alpha)}\frac{\Gamma(1+i\alpha)}{\Gamma(N+i\alpha)}\left[\sum_{s\neq n}^{N-1}\frac{1}{s-i\alpha}+\sum_{s\neq n}^{N-1}\frac{1}{s+i\alpha}-2\sum_{s=1}^{N-1}\frac{1}{s+1}+\left\{\frac{2n}{\alpha^2+n^2}\right\}\right]. \tag{A41}$$

The total residue at $\alpha=in$ from Eq. (A41) is

$$\frac{in(-1)^n e^{N\beta\lambda nz}}{\Gamma(N-n)\Gamma(N+n)}\left[2\sum_{s\neq n}^{N-1}\left(\frac{s}{s^2-n^2}\right)-2\sum_{s=1}^{N-1}\frac{1}{s+1}+\frac{1}{2n}+[\Psi(N+n)-\Psi(N-n)]-\frac{1+N\beta\lambda nz}{n}\right]$$
$$=\frac{-i\exp(N\beta\lambda nz)}{[\Gamma(N)]^2}\left(C_n^N-\frac{1}{n}A_n^N(1+N\beta\lambda nz)\right), \tag{A42}$$

provided $z<0$ (otherwise it vanishes) and so

$$\frac{1}{N}\sum_{n=1}^{N}\int\frac{dk}{2\pi}\int d\mathbf{u}\exp\left(-ikz-\lambda\beta\sum_{l=1}^{N-1}(C_l+i\alpha D_{nl})u_l\right)\left(\sum_{l=1}^{n-1}lu_l+\sum_{l=n}^{N-1}(N-l)u_l\right)$$

$$=\frac{2N\beta\pi Gm^2}{(2\pi G\beta m^2)^N[\Gamma(N)]^2}\sum_{l=1}^{N-1}\left[C_l^N-\frac{1}{l}A_l^N[1-2N(\beta\pi Gm^2)l|z|]\right]\exp[-2N(\beta\pi Gm^2)l|z|],\tag{A43}$$

where

$$C_n^N=\frac{n(-1)^{n+1}[\Gamma(N)]^2}{\Gamma(N-n)\Gamma(N+n)}\left[2\sum_{s\neq n}^{N-1}\left(\frac{s}{s^2-n^2}\right)-2\sum_{s=1}^{N-1}\frac{1}{s+1}+\frac{1}{2n}+[\Psi(N+n)-\Psi(N-n)]\right].\tag{A44}$$

Finally, we consider the expression

$$\frac{1}{N}\sum_{n=1}^{N}\int d\mathbf{u}\exp\left(-\lambda\beta\sum_{l=1}^{N-1}(C_l+i\alpha D_{nl})u_l\right)\sum_{s=1}^{N-1}\sum_{t=s+1}^{N-1}(N-t)(t-s)su_su_t$$

$$=\frac{1}{N}\sum_{n=1}^{N}\sum_{s=1}^{N-1}\sum_{t=s+1}^{N-1}(N-t)(t-s)sI_{n,s,t}^2$$

$$=\frac{1}{(2\pi G\beta m^2)^{N+1}}\frac{1}{N}\sum_{n=1}^{N}\sum_{s=1}^{N-1}\sum_{t=s+1}^{N-1}\frac{(t-s)s}{C_s+i\alpha D_{n,s}}\frac{(N-t)}{C_t+i\alpha D_{nt}}\prod_{l=1}^{N-1}\frac{l}{C_l+i\alpha D_{nl}}.\tag{A45}$$

Interchanging the order of the sums gives

$$\sum_{n=1}^{N}\sum_{s=1}^{N-1}\sum_{t=s+1}^{N-1}\frac{(t-s)s}{C_s+i\alpha D_{ns}}\frac{(N-t)}{C_t+i\alpha D_{nt}}\prod_{l=1}^{N-1}\frac{1}{C_l+i\alpha D_{nl}}$$

$$=\sum_{s=1}^{N-1}\sum_{t=s+1}^{N-1}\sum_{n=1}^{N}\left(\frac{(t-s)s}{C_s+i\alpha D_{ns}}\frac{(N-t)}{C_l+i\alpha D_{nt}}\frac{[\Gamma(N)]^2}{(n-1)!(N-n)!}\frac{\Gamma(N-n+1-i\alpha)}{\Gamma(N-i\alpha)}\frac{\Gamma(n+i\alpha)}{\Gamma(N+i\alpha)}\right)$$

$$=\frac{[\Gamma(N)]^2}{\Gamma(N+i\alpha)\Gamma(N-i\alpha)}\sum_{s=1}^{N-1}\sum_{t=s+1}^{N-1}\left[\sum_{n=1}^{s}\left(\frac{(t-s)s}{(N-s)(t+i\alpha)(s+i\alpha)}\frac{\Gamma(N-n+1-i\alpha)}{\Gamma(N-n+1)}\frac{\Gamma(n+i\alpha)}{\Gamma(n)}\right)\right.$$

$$+\sum_{n=s+1}^{t}\left(\frac{(t-s)}{(N-s-i\alpha)(t+i\alpha)}\frac{\Gamma(N-n+1-i\alpha)}{\Gamma(N-n+1)}\frac{\Gamma(n+i\alpha)}{\Gamma(n)}\right)$$

$$\left.+\sum_{n=t+1}^{N}\left(\frac{(N-t)(t-s)}{t(N-s-i\alpha)(N-t-i\alpha)}\frac{\Gamma(N-n+1-i\alpha)}{\Gamma(N-n+1)}\frac{\Gamma(n+i\alpha)}{\Gamma(n)}\right)\right].\tag{A46}$$

When summed over n, we obtain

$$\frac{1}{N}\sum_{n=1}^{N}\sum_{s=1}^{N-1}\sum_{t=s+1}^{N-1}\frac{(t-s)s}{C_s+i\alpha D_{ns}}\frac{(N-t)}{C_t+i\alpha D_{nt}}\prod_{l=1}^{N-1}\frac{1}{C_l+i\alpha D_{nl}}=\frac{\sum_{k=1}^{N-1}a_k\alpha^{2k}}{\prod_{l=1}^{N-1}(\alpha^2+l^2)^2},\tag{A47}$$

which has double poles and single-pole residues at $\alpha=\pm in$ for every nonzero value of $n<N$. The coefficients of the polynomial in the numerator are calculable, but we have not found any closed-form expression for them. The table below contains results for values up to $N=10$.

N	$\sum_{k=1}^{N-1} a_k \alpha^{2k}$
3	$2(a^2-1)(a^2-2)$
4	$4(180-109a^2+10a^4+11a^6)$
5	$8(36576-3820a^2-75a^4+1590a^6+149a^8)$
6	$48(5263200+1132124a^2+162455a^4+170877a^6+25445a^8+899a^{10})$
7	$288(1455926400+635262768a^2+123441248a^4+43494899a^6+6982689a^8+399833a^{10}+7163a^{12})$
8	$1152(1067349830400+638760596688a^2+149678407480a^4+34350170141a^6+5098185940a^8+351491854a^{10}+10505180a^{12}+110317a^{14})$
9	$41472(143590977331200+103719257351424a^2+27508922447056a^4+5307728339928a^6+709200726957a^8+52255261172a^{10}+1966842318a^{12}+35272476a^{14}+237469a^{16})$
10	$414720(108807366682828800+89224919703007488a^2+25843463092699920a^4+4662765631167688a^6+573351004521465a^8+42828568506933a^{10}+1816855681610a^{12}+42140421618a^{14}+493478205a^{16}+2266273a^{18})$.

Hence we obtain

$$\frac{1}{N}\sum_{n=1}^{N}\int d\mathbf{u}\exp\left(-\lambda\beta\sum_{l=1}^{N-1}(C_l+i\alpha D_{nl})u_l\right)\sum_{s=1}^{N-1}\sum_{t=s+1}^{N-1}(N-1)(t-s)su_su_t$$
$$=\frac{2\times 2N\beta\pi Gm^2}{(2\pi G\beta m^2)^{N+1}[\Gamma(N)]^2}\sum_{l=1}^{N-1}\{D_l^N+K_l^N[1-2N(\beta\pi Gm^2)l|z|]\}\exp[-2N(\beta\pi Gm^2)l|z|], \tag{A48}$$

where the coefficients D_l^N and K_l^N are determined by the residues given above. These are given in the next two tables.

D_l^N	$N=3$	$N=4$	$N=5$	$N=6$	$N=7$	$N=8$	$N=9$	$N=10$
$l=1$	$-\frac{1}{3}$	$-\frac{1}{4}$	$-\frac{2}{45}$	$\frac{31}{144}$	$\frac{2117}{4200}$	$-\frac{2917}{3600}$	$\frac{24869}{22050}$	$\frac{113931}{78400}$
$l=2$	$\frac{11}{12}$	$\frac{209}{125}$	$\frac{169}{90}$	$\frac{5599}{3087}$	$-\frac{2131}{1344}$	$\frac{30347}{24300}$	$\frac{941}{1125}$	$\frac{601906}{1630475}$
$l=3$		$-\frac{1387}{1500}$	$-\frac{9278}{5145}$	$-\frac{12781}{5488}$	$-\frac{1468}{567}$	$-\frac{85427}{32400}$	$-\frac{836537}{332750}$	$-\frac{3601223}{1597200}$
$l=4$			$\frac{72917}{123480}$	$-\frac{431581}{333396}$	$\frac{214607}{113400}$	$\frac{18987467}{8085285}$	$\frac{3170471}{1197900}$	$\frac{407903579}{146210350}$
$l=5$				$-\frac{1005251}{3333960}$	$-\frac{22515781}{30187080}$	$-\frac{31621187}{25874640}$	$-\frac{440479867}{263178630}$	$-\frac{4721124703}{2292578288}$
$l=6$					$\frac{162182479}{1207483200}$	$\frac{8830831883}{23686076700}$	$\frac{2430681577}{3582153575}$	$\frac{10937730746}{10746460725}$
$l=7$						$-\frac{108510757181}{1989630442800}$	$-\frac{7762820713}{46056260250}$	$-\frac{22140852323}{65502236800}$
$l=8$							$\frac{53405900137}{2579150574000}$	$\frac{9952938913257}{140792964111800}$
$l=9$								$-\frac{151474036840183}{20274186832099200}$

K_l^N	$N=3$	$N=4$	$N=5$	$N=6$	$N=7$	$N=8$	$N=9$	$N=10$
$l=1$	$-\frac{1}{3}$	$-\frac{1}{2}$	$-\frac{3}{5}$	$-\frac{2}{3}$	$-\frac{5}{7}$	$-\frac{3}{4}$	$-\frac{7}{9}$	$-\frac{4}{5}$
$l=2$	$-\frac{5}{24}$	$-\frac{1}{25}$	$\frac{1}{10}$	$\frac{31}{147}$	$\frac{67}{224}$	$\frac{10}{27}$	$\frac{193}{450}$	$\frac{289}{605}$
$l=3$		$\frac{7}{50}$	$-\frac{27}{245}$	$\frac{43}{784}$	$-\frac{11}{1512}$	$-\frac{37}{540}$	$-\frac{381}{3025}$	$-\frac{1297}{7260}$
$l=4$			$-\frac{257}{3920}$	$-\frac{829}{10584}$	$-\frac{361}{5040}$	$-\frac{853}{16335}$	$-\frac{551}{21780}$	$\frac{1153}{204490}$
$l=5$				$-\frac{2761}{10584}$	$\frac{12431}{304920}$	$\frac{3163}{65340}$	$\frac{44893}{920205}$	$\frac{24587}{572572}$
$l=6$					$-\frac{34541}{3659040}$	$-\frac{50012}{2760615}$	$-\frac{368191}{1431430}$	$-\frac{66629}{2147145}$
$l=7$						$\frac{248029}{77297220}$	$\frac{234797}{32207175}$	$\frac{274399}{22902880}$
$l=8$							$-\frac{1075190}{1030629600}$	$-\frac{1812235}{661893232}$
$l=9$								$\frac{6514549}{19856796960}$

Summarizing:

$$\frac{1}{N}\sum_{n=1}^{N}\int\frac{dk}{2\pi}\int d\mathbf{u}\exp\left(-ikz-\lambda\beta\sum_{l=1}^{N-1}(C_l+i\alpha D_{nl})u_l\right)=\frac{2N\beta\pi Gm^2}{(2\pi G\beta m^2)^{N-1}[\Gamma(N)]^2}\sum_{l=1}^{N-1}A_l^N\exp[-2N(\beta\pi Gm^2)l|z|] \tag{A49}$$

$$\frac{1}{N}\sum_{n=1}^{N}\int\frac{dk}{2\pi}\int d\mathbf{u}\left(\sum_{s=1}^{N-1}s(N-s)u_s\right)\exp\left(-ikz-\lambda\beta\sum_{l=1}^{N-1}(C_l+i\alpha D_{nl})u_l\right),$$

$$=\frac{2N\beta\pi Gm^2}{(2\pi G\beta m^2)^{N}[\Gamma(N)]^2}\sum_{l=1}^{N-1}\{B_l^N-A_l^N[1-2N(\beta\pi Gm^2)l|z|]\}\exp[-2N(\beta\pi Gm^2)L|z|] \tag{A50}$$

$$\frac{1}{N}\sum_{n=1}^{N}\int\frac{dk}{2\pi}\int d\mathbf{u}\left(\sum_{s=1}^{l-1}sI_{n,s}^1+\sum_{s=n}^{N-1}(N-s)I_{n,s}^1\right)\exp\left(-ikz-\lambda\beta\sum_{l=1}^{N-1}(C_l+i\alpha D_{nl})u_l\right)$$

$$=\frac{2N\beta\pi Gm^2}{(2\pi G\beta m^2)^{N}[\Gamma(N)]^2}\sum_{l=1}^{N-1}\left[C_l^N-\frac{1}{l}A_l^N[1-2N(\beta\pi Gm^2)l|z|]\right]\exp[-2N(\beta\pi Gm^2)l|z|], \tag{A51}$$

$$\frac{1}{N}\sum_{n=1}^{N}\int\frac{dk}{2\pi}\int d\mathbf{u}\left(\sum_{s=1}^{N-1}\sum_{t=s+1}^{N-1}(N-t)(t-s)su_su_t\right)\exp\left(-ikz-\lambda\beta\sum_{l=1}^{N-1}(C_l+i\alpha D_{nl})u_l\right)$$

$$=\frac{2N\beta\pi Gm^2}{(2\pi G\beta m^2)^{N+1}[\Gamma(N)]^2}\sum_{l=1}^{N-1}\{D_l^N+K_l^N[1-2N(\beta\pi Gm^2)l|z|]\}\exp[-2N(\beta\pi Gm^2)l|z|]. \tag{A52}$$

The final expression for the one-particle distribution function is

$$f_{cn}(p,z)=\sqrt{N}(\sqrt{2\pi}G/c^3)^{(N-1)}[(N-1)!]^2\left(\frac{2\pi m}{\beta}\right)^{(N-2)/2}\frac{2N\beta\pi m^2(\beta mc^2)^{3(N-1)/2}}{\sqrt{N-1}(2\pi G\beta m^2)^{N-1}[\Gamma(N)]^2}$$

$$\times\exp\left[-\frac{1}{\beta mc^2}\left\{\frac{3(N-1)^2}{8N}\lambda_1-\frac{(N-1)[\lambda_2(N-1)+\lambda_3]}{N}-\lambda_4\sum_{k=1}^{N-1}\sum_{l=k+1}^{N-1}\frac{(l-k)}{l(N-k)}\right\}\right]$$

$$\times\sum_{l=1}^{N-1}\left[\left\{A_l^N\left[1+\frac{\lambda_1}{2\beta mc^2}\left(\frac{\beta^2p^4[1+(N-1)^3]}{(2m)^2(N-1)^3}+\frac{3(N-2)\beta p^2}{2m(N-1)^2}+\frac{3(N-2)^2}{4(N-1)}\right)\right]\right.\right.$$

$$+\frac{1}{\beta mc^2}\left(\frac{(\lambda_3-\lambda_2)\beta p^2}{m(N-1)^2}-\frac{[\lambda_3+(N-2)\lambda_2]}{(N-1)}\right)\{B_l^N-A_l^N[1-2N(\beta\pi Gm^2)l|z|]\}$$

$$+\frac{2[2\lambda_3+(N-2)\lambda_2]}{\beta mc^2}\left(\frac{1}{2(N-1)}-\frac{N\beta p^2}{2m}\left[\frac{1}{(N-1)^2}\right]\right)\left[C_l^N-\frac{1}{l}A_l^N[1-2N(\beta\pi Gm^2)l|z|]\right]$$

$$\left.\left.-\frac{\lambda_4}{\beta mc^2}\{D_l^N+K_l^N[1-2N(\beta\pi Gm^2)l|z|]\}\right\}\exp\left(-\frac{N\beta p^2}{2m(N-1)}-2\pi GN\beta m^2l|z|\right)\right]$$

$$=\frac{(2\pi Gm^2)(N\beta)^{3/2}}{\sqrt{2\pi m(N-1)}}\exp\left[-\frac{1}{\beta mc^2}\left\{\frac{3(N-1)^2}{8N}\lambda_1-\frac{(N-1)[\lambda_2(N-1)+\lambda_3]}{N}-\lambda_4\sum_{k=1}^{N-1}\sum_{l=k+1}^{N-1}\frac{(l-k)}{l(N-k)}\right\}\right]$$

$$\times\sum_{l=1}^{N-1}\left[\left\{A_l^N\left[1+\frac{\lambda_1}{2\beta mc^2}\left(\frac{\beta^2p^4[1+(N-1)^3]}{(2m)^2(N-1)^3}+\frac{3(N-2)\beta p^2}{2m(N-1)^2}+\frac{3(N-2)^2}{4(N-1)}\right)\right]\right.\right.$$

$$+\frac{1}{\beta mc^2}\left(\frac{(\lambda_3-\lambda_2)\beta p^2}{m(N-1)^2}-\frac{[\lambda_3+(N-2)\lambda_2]}{(N-1)}\right)\{B_l^N-A_l^N[1-2N(\beta\pi Gm^2)l|z|]\}$$

$$+\frac{2[2\lambda_3+(N-2)\lambda_2]}{\beta mc^2}\left(\frac{1}{2(N-1)}-\frac{N\beta p^2}{2m}\left[\frac{1}{(N-1)^2}\right]\right)\left[C_l^N-\frac{1}{l}A_l^N[1-2N(\beta\pi Gm^2)l|z|]\right]$$

$$\left.\left.-\frac{\lambda_4}{\beta mc^2}\{D_l^N+K_l^N[1-2N(\beta\pi Gm^2)l|z|]\}\right\}\exp\left(-\frac{N\beta p^2}{2m(N-1)}-2\pi GN\beta m^2l|z|\right)\right]. \quad \text{(A53)}$$

The canonical density distribution function is given by integration of $f_{cn}(p,z)$ over p,

$$\rho_c(z)=\int_{-\infty}^{\infty}dp\,f_{cm}(p,z)$$

$$=\frac{(2\pi Gm^2)(N\beta)^{3/2}}{\sqrt{2\pi m(N-1)}}\sqrt{\frac{2\pi m(N-1)}{N\beta}}$$

$$\times\exp\left[-\frac{1}{\beta mc^2}\left\{\frac{3(N-1)^2}{8N}\lambda_1-\frac{(N-1)[\lambda_2(N-1)+\lambda_3]}{N}-\lambda_4\sum_{k=1}^{N-1}\sum_{l=k+1}^{N-1}\frac{(l-k)}{l(N-k)}\right\}\right]$$

$$\times\sum_{l=1}^{N-1}\left[\left\{A_l^N\left[1+\frac{\lambda_1}{2\beta mc^2}\left(\frac{3[1+(N-1)^3]}{4N^2(N-1)}+\frac{3(N-2)}{2N(N-1)}+\frac{3(N-2)^2}{4(N-1)}\right)\right]\right.\right.$$

$$+\frac{1}{\beta mc^2}\left(\frac{(\lambda_3-\lambda_2)}{N(N-1)}-\frac{[\lambda_3+(N-2)\lambda_2]}{(N-1)}\right)\{B_l^N-A_l^N[1-2N(\beta\pi Gm^2)l|z|]\}$$

$$+\frac{2[2\lambda_3+(N-2)\lambda_2]\pi Gm}{\beta mc^2}\left(\frac{1}{2(N-1)}-\left[\frac{1}{2(N-1)^2}\right]\right)\left[C_l^N-\frac{1}{l}A_l^N[1-2N(\beta\pi Gm^2)l|z|]\right]$$

$$\left.-\frac{\lambda_4}{\beta mc^2}\{D_l^N+K_l^N[1-2N(\beta\pi Gm^2)l|z|]\}\exp(-2\pi GN\beta m^2 l|z|)\right]$$

$$=(2\pi Gm^2N\beta)\exp\left[-\frac{1}{\beta mc^2}\left\{\frac{3(N-1)^2}{8N}\lambda_1-\frac{(N-1)[\lambda_2(N-1)+\lambda_3]}{N}-\lambda_4\sum_{k=1}^{N-1}\sum_{l=k+1}^{N-1}\frac{l(N-k)}{L(N-k)}\right\}\right]$$

$$\times\sum_{l=1}^{N-1}\left\{A_l^N+\frac{1}{\beta mc^2}\left[\frac{3\lambda_1}{8}\frac{(N-1)^2}{N}A_l^N-\left(\frac{[\lambda_2(N-1)+\lambda_3]}{N}\right)B_l^N-\lambda_4D_l^N\right]\right.$$

$$\left.+\frac{1}{\beta mc^2}\left[\left(\frac{[\lambda_2(N-1)+\lambda_3]}{N}\right)A_l^N-\lambda_4K_l^N\right][1-2N(\beta\pi Gm^2)l|z|]\right\}\exp(-2\pi Gm^2N\beta l|z|). \tag{A54}$$

The final expressions have all $\lambda_i=1$. It is straightforward to show that the coefficients A_l^N and B_l^N obey the sum rules

$$\sum_{l=1}^{N-1}A_l^N=\frac{1}{2}\frac{N-1}{2N-3}, \tag{A55}$$

$$\sum_{l=1}^{N-1}B_l^N=\frac{1}{2}\frac{(N-1)^2}{2N-3}, \tag{A56}$$

where Eq. (A55) was previously derived in the $c\to\infty$ limit [1]. Since $\int_{-\infty}^{\infty}dz\,\rho_c(z)=1$, we must have

$$2\sum_{l=1}^{N-1}\frac{1}{l}\left\{A_l^N+\frac{1}{\beta mc^2}\left[\frac{3\lambda_1}{8}\frac{(N-1)^2}{N}A_l^N-\frac{2}{2}\left(\frac{[\lambda_2(N-1)+\lambda_3]}{N}\right)B_l^N-\lambda_4D_l^N\right]\right\}$$

$$=\exp\left[\frac{1}{\beta mc^2}\left\{\frac{3(N-1)^2}{8N}\lambda_1-\frac{(N-1)[\lambda_2(N-1)+\lambda_3]}{N}-\lambda_4\sum_{k=1}^{N-1}\sum_{l=k+1}^{N-1}\frac{(l-k)}{l(N-k)}\right\}\right], \tag{A57}$$

or alternatively, to the relevant order in c,

$$\sum_{l=1}^{N-1}\frac{1}{l}A_l^N=\frac{1}{2}, \tag{A58}$$

$$\sum_{l=1}^{N-1}\frac{1}{l}B_l^N=\frac{1}{2}(N-1), \tag{A59}$$

$$\sum_{l=1}^{N-1}\frac{1}{l}D_l^N=\frac{1}{2}\sum_{k=1}^{N-1}\sum_{l=k+1}^{N-1}\frac{(l-k)}{l(N-k)}, \tag{A60}$$

each of which can be straightforwardly verified. We also can show

$$\sum_{l=1}^{N-1}\frac{1}{l}C_l^N=C_1^N=\frac{(N-1)}{N}, \tag{A61}$$

$$\sum_{l=1}^{\infty}C_l^N=\frac{\pi^2}{12}. \tag{A62}$$

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